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Abstract

In this paper, we study the transformed Rayleigh Lomax (Trans-RL) distribution which belongs to a certain family of two parameters lifetime distributions given by Wang et al. (2010). Confidence intervals and inverse estimators of the Trans-RL parameters are derived in terms of the order statistics. A simulation study is conducted to report the coverage probabilities, the average biases and the average relative mean square errors for the maximum likelihood, L-moments and inverse estimators. We compare the performance of these methods under different schemes of progressively Type-II right censoring. Finally, an illustrative example is provided to demonstrate the proposed methods.

Keywords: L-moments; Confidence intervals; Inverse estimators; Order Statistics, Proportional hazard family.

1 Introduction

Censoring is one of the useful sampling techniques in life test experiment which is used to save time and cost of testing units, see (Lawless, 2011) and (Meeker and Escobar, 2014). Type-I and Type-II censoring are the two most common censoring schemes, where in Type-I censoring scheme, the total duration of the study is fixed and the number of failures is random, whereas under censoring of Type-II, the number of failures is fixed in advance and the total duration of the study is random.

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One of the main extensions of Type-II censoring scheme is progressive Type-II right censoring which has received great attentions. Herd (1956) was the first to study estimation of the population parameters based on progressively censored samples. Cohen (1963) studied the importance of progressive censoring in life-testing reliability experiments. Balakrishnan and Aggarwala (2000) provided details about right progressive censoring in theories and applications.

Let $Y_1, Y_2, ..., Y_n$ be independent and identically distributed (i.i.d) random lifetimes of n units. A progressive Type-II right censored sample can be obtained in the following way: Suppose that n units are placed on life test at time zero. Before we begin the test, a number $m(< n)$ is fixed and the censoring scheme $R = (R_1, ..., R_m)$ with $R_j \geq 0$ and $\sum_{j=1}^{m} R_j + m = n$ is specified. Immediately following the first failure, R_1 surviving units are removed from the test at random. Then, immediately following the second observed failure, R_2 surviving units are removed from the test at random . This process continues until at the time of the $m-th$ observed failure, the remaining $R_m = n - R_1 - R_2 - ... - R_{m-1} - m$ units are removed from the experiment and censored. This censoring scheme includes as special cases the complete sample (when $m = n$ and $R_1 = ... = R_m = 0$).

Several lifetime distributions associated with censored sampling are available and have wide applications in engineering, science, public health and medicine. For example, the one parameter exponential distribution under censored sampling has received great attention in the literature. See (Ehsan Saleh, 1967), (Pettitt, 1977), (Wright et al., 1978) and (Sundberg, 2001).

According to Marshall and Olkin (2007), let G be a distribution function depending only on the shape parameter α with hazard rate $R = -log(1 - G)$. Suppose that $F(.; \alpha, \sigma)$ is defined by the formula

$$F(y; \alpha, \sigma) = 1 - [1 - G(y; \alpha)]^{\sigma}, \quad (1)$$

Then, σ is called a frailty parameter and $\{F(.; \alpha, \sigma); \alpha, \sigma > 0\}$ is a frailty parameter family, or alternatively, a proportional hazard family. The Weibull distribution, the Gompertz distribution and the Lomax distribution are examples included in family (1). In this paper, we study the inference under progressively Type-II right censored sampling for a new distribution belongs to the family (1) called the transformed Rayleigh Lomax (Trans-RL) distribution. The rest of the paper is organized as follows. In section 2, a new distribution named the Rayleigh Lomax (RL) distribution is defined, and the definition of the transformed-RL distribution is followed by. Under progressively Type-II right censoring, the confidence intervals as well as the inverse estimation for the Trans-RL parameters are studied in section 3. In section 4, the coverage probabilities, the average relative biases and average relative mean square errors for the MLE, the method of L-moments and inverse estimators are calculated for different progressive censoring schemes through a simulation study. Finally, a real dataset is provided to illustrate the proposed method.

2 The Rayleigh-Lomax Distribution

Lomax distribution is one of the well known distributions that is very useful in many fields such as engineering, reliability and life testing. However, this distribution does not provide great flexibility in modeling data. Thus, Lomax distribution can be generalized by presenting additional parameters such as shape, scale or location in the distribution and then observing the characteristic of the new distribution. Several generalized classes of distributions are available such as exponentiated Lomax (Abdul-Moniem and Abdel-Hameed, 2012), Beta-Lomax((Rajab et al., 2013), exponential Lomax distribution (El-Bassiouny et al., 2015) and Gumbel-Lomax (Tahir et al., 2016).

El-Bassiouny et al. (2015) proposed a new generalization of Lomax distribution by adding a scale parameter $\beta > 0$ to the Lomax distribution. Let $G(x)$ denotes the cumulative density function (cdf) of Lomax distribution and $f(t)$ is the probability density function (pdf) of the exponential distribution. Then the cdf for the exponential Lomax (ELomax) distribution is given by the general expression

$$F(x;\alpha,\lambda,\beta) = \int_0^{\frac{1}{1-G(x;\alpha,\lambda)}} f(t;\beta)dt. \tag{2}$$

Therefore, the ELomax cdf is given by

$$\int_0^{\frac{1}{\left(\frac{\lambda}{x+\lambda}\right)^\alpha}} \beta e^{-\beta t} dt = 1 - e^{-\beta\left(\frac{\lambda}{x+\lambda}\right)^{-\alpha}}, \alpha, \lambda, \beta > 0. \tag{3}$$

Rayleigh Lomax (RL) distribution is another extension of Lomax distribution which provides great fit in modeling wide range of real data sets. It is a very flexible distribution that by changing its parameters, some different useful distributions can be obtained. The RL distribution is defined using the general expression in (2) as

$$F(x;\alpha,\lambda,\sigma) = \int_0^{\frac{1}{\left(\frac{\lambda}{x+\lambda}\right)^\alpha}} f(t;\sigma)dt, \tag{4}$$

where $f(t;\sigma)$ is the pdf of Rayleigh distribution given by

$$f(t;\sigma) = \frac{t}{\sigma^2} e^{-\frac{t^2}{2\sigma^2}}, \quad t \geq 0, \sigma > 0. \tag{5}$$

From (4), The Rayleigh Lomax cdf is given by

$$F(x;\alpha,\lambda,\sigma) = 1 - e^{\frac{-1}{2\sigma^2}\left(\frac{x+\lambda}{\lambda}\right)^{2\alpha}}, \tag{6}$$

The corresponding pdf is given by taking the derivative of (6)

$$f(x;\alpha,\lambda,\sigma) = \frac{\alpha}{\lambda\sigma^2}\left(\frac{x+\lambda}{\lambda}\right)^{2\alpha-1} e^{\frac{-1}{2\sigma^2}\left(\frac{x+\lambda}{\lambda}\right)^{2\alpha}}, \quad x > -\lambda, \quad \alpha, \lambda, \sigma > 0, \tag{7}$$

where α is the shape parameter and λ and σ are scale parameters of Rayleigh Lomax distribution. Rayleigh Lomax distribution behaves as a lifetime distribution by adding the scale parameter λ to the RL random variable X. Particularly, let $Y = \frac{X+\lambda}{\lambda}$, then the pdf of the transformed RL distribution is

$$f_Y(y;\alpha,\sigma) = \frac{\alpha}{\sigma^2} y^{2\alpha-1} e^{\frac{-1}{2\sigma^2} y^{2\alpha}}, \quad y,\alpha,\sigma > 0, \tag{8}$$

and its cdf is given by

$$F_Y(y;\alpha,\sigma) = 1 - e^{\frac{-1}{2\sigma^2} y^{2\alpha}}. \tag{9}$$

Regarding the Trans-RL distribution, (1) can be written as

$$F(y;\alpha,\sigma) = 1 - [1 - G(y;\alpha)]^{\frac{1}{2\sigma^2}}, \quad y,\alpha,\sigma > 0, \tag{10}$$

where $G(y;\alpha) = 1 - e^{-y^{2\alpha}}$.

3 Confidence Intervals and Inverse Estimation of Parameters α and σ

The following results, which are also stated in Wang et al. (2010), are needed

Theorem 1. *(I) If $V_{i:m:n} = -log(1-F(Y_{i:m:n};\alpha,\sigma)), \quad i = 1,...,m$, then $V_{1:m:n},...,V_{m:m:n}$ is a progressively type II right censored sample from the standard exponential distribution with sample size n and censoring scheme $R = (R_1,...,R_m)$. In the Trans-RL distribution, $V_{i:m:n} = \frac{1}{2\sigma^2} Y_{i:m:n}^{2\alpha}$.*
(II) If $W_1 = nV_{1:m:n}$, $W_i = \left[n - \sum_{j=1}^{i-1}(R_j+1)\right](V_{i:m:n} - V_{i-1:m:n})$, $i = 2,...,m$, then $W_1,...,W_m$ are independent standard exponential random variates.
(III) If $S_i = \sum_{j=1}^{i} W_j, i = 1,...,m$ and $U_{(i)} = \frac{S_i}{S_m}, i = 1,...,m-1$, then $U_{(1)} < ... < U_{(m-1)}$ are order statistics from the uniform(0,1) distribution with sample size m-1.

Proof. (I) Balakrishnan and Aggarwala (2000) provided the joint pdf of all m progressively type II right censored order statistics as follows.

$$f_{Y_{1:m:n},...,Y_{m:m:n}}(y_1,...,y_m) = c\prod_{i=1}^{m} f(y_i)[1-F(y_i)]^{R_i}, \quad y_1 < ... < y_m \tag{11}$$

where $c = n(n-R_1-1)...(n-R_1-R_2-...-R_{m-1}-m+1)$. For the Trans-RL distribution, the joint pdf of all m progressively Type-II right censored order statistics is

$$f_{Y_{1:m:n},...,Y_{m:m:n}}(y_1,...,y_m) = c\prod_{i=1}^{m} \frac{\alpha}{\sigma^2} y_i^{2\alpha-1}\left(e^{\frac{-1}{2\sigma^2} y_i^{2\alpha}}\right)^{R_i+1}. \tag{12}$$

Then it is easy for us to obtain

$$f_{V_{1:m:n},...,V_{m:m:n}}(v_1,...,v_m) = ce^{-\sum_{i=1}^{m}(R_i+1)v_i}, \quad 0 < v_1 < ... < v_m < \infty, \tag{13}$$

which implies that $V_{1:m:n}, ..., V_{m:m:n}$ is a progressively Type-II right censored sample from the standard exponential distribution.

(II) If

$$\begin{aligned} W_1 &= nV_{1:m:n}, \\ W_2 &= (n - R_1 - 1)(V_{2:m:n} - V_{1:m:n}), \\ &\vdots \\ W_m &= (n - R_1 - ... - R_{m-1} - m + 1)(V_{m:m:n} - V_{m-1:m:n}). \end{aligned}$$

Then,

$$\begin{aligned} V_{1:m:n} &= \frac{W_1}{n}, \\ V_{2:m:n} &= \frac{W_1}{n} + \frac{W_2}{n - R_1 - 1}, \\ &\vdots \\ V_{m:m:n} &= \frac{W_1}{n} + ... + \frac{W_m}{n - R_1 - ... - R_{m-1} - m + 1}. \end{aligned}$$

Hence, from (13), we have

$$\begin{aligned} f_{W_1,...,W_m}(w_1,...,w_m) &= e^{-(R_1+1)\frac{w_1}{n}}.e^{-(R_2+1)\left(\frac{w_1}{n}+\frac{w_2}{n-R_1-1}\right)}...e^{-(R_m+1)\left(\frac{w_1}{n}+...+\frac{w_m}{n-R_1-...-R_m-(m-1)}\right)} \\ &= e^{-\sum_{i=1}^{m}(R_i+1)\frac{w_1}{n}}.e^{-\sum_{i=2}^{m}(R_i+1)\frac{w_2}{n-R_1-1}}...e^{-(R_m+1)\frac{w_m}{n-R_1-...-R_{m-1}-(m-1)}}. \\ &= e^{-\sum_{i=1}^{m} w_i}, \quad w_i \geq 0. \end{aligned} \tag{14}$$

Therefore, $W_1, ..., W_m$ are independent standard exponential random variates.

(III) The probability distribution function of the order statistic U_i, $i = 1, ..., m-1$. from the uniform(0,1) is given by

$$\begin{aligned} f_{U_{(i)}}(u) &= \frac{(m-1)!}{(i-1)!(m-i-1)!}u^{i-1}(1-u)^{m-i-1}, \\ &= \frac{\Gamma(m)}{\Gamma(i)\Gamma(m-i)}u^{i-1}(1-u)^{m-i-1}, \quad 0 < u_{(1)} < u_{(2)} < ... < u_{(m-1)} < 1 \end{aligned} \tag{15}$$

which implies that $U_{(i)} \sim Beta(i, m-i)$.

Given $S_i = \sum_{j=1}^{i} W_j, \quad i = 1, ..., m$, we need to show that $U_i = \frac{S_i}{S_m}, \quad i = 1, ..., m-1$ is order statistic from the uniform (0,1) distribution with sample size $m-1$. Since $W_1, ..., W_m$ are independent standard exponential random variates, then $S_i = \sum_{j=1}^{i} W_j$ is a random variable from the gamma distribution with the shape parameter $i > 0$ and the rate parameter equals 1. Let $S_{i+1} = W_{i+1} + W_{i+2} + ... + W_m$ and $S_m = S_i + S_{i+1}$, where S_{i+1} follows the gamma distribution with the shape parameter $m-i > 0$ and the rate parameter equals 1.

Since S_i and S_{i+1} are independent random variables, the joint pdf of of S_i and S_{i+1} is given as

$$f(s_i, s_{i+1}) = \frac{s_i^{i-1}}{\Gamma(i)} e^{-s_i} . \frac{1}{\Gamma(m-i)} s_{i+1}^{m-i-1} e^{-s_{i+1}}, \quad S_i, S_{i+1} \geq 0, i, m > 0. \tag{16}$$

Let $K = \frac{S_i}{S_i + S_{i+1}}$ and $S_m = S_i + S_{i+1}$, then $S_i = KS_m$ and $S_{i+1} = S_m(1-K)$ with

$$\frac{\partial(S_i, S_{i+1})}{\partial(S_m, K)} = \begin{vmatrix} K & S_m \\ 1-K & -S_m \end{vmatrix} = S_m.$$

Then, the joint pdf of K and S_m is given as

$$f(k, s_m) = \frac{\Gamma(m)}{\Gamma(i)\Gamma(m-i)} k^{i-1} (1-k)^{m-i-1} \frac{s_m^{m-1}}{\Gamma(m)} e^{-s_m}. \tag{17}$$

By factorization theorem, $\frac{S_i}{S_i + S_{i+1}} = \frac{S_i}{S_m} \sim Beta(i, m-i)$.

Therefore, $U_{(i)} = \frac{S_i}{S_m}, i = 1, ..., m-1$, where $U_{(1)} < ... < U_{(m-1)}$ are order statistics from the uniform(0,1) distribution with sample size $m-1$.

3.1 Interval estimation of parameter α

In order to construct the confidence interval of the parameter α, we consider the following pivotal quantity

$$\begin{aligned} W(2\alpha) &= \sum_{i=1}^{m-1} (-2log(U_{(i)})) = 2 \sum_{i=1}^{m-1} log\left(\frac{S_m}{S_i}\right) \\ &= 2 \sum_{i=1}^{m-1} log\left[\frac{\sum_{j=1}^{m} (R_j+1) V_{j:m:n}}{\sum_{j=1}^{i} (R_j+1) V_{j:m:n} + [n - \sum_{j=1}^{i} (R_j+1)] V_{i:m:n}}\right] \\ &= 2 \sum_{i=1}^{m-1} log\left[\frac{\sum_{j=1}^{m} (R_j+1) Y_{j:m:n}^{2\alpha}}{\sum_{j=1}^{i} (R_j+1) Y_{j:m:n}^{2\alpha} + [n - \sum_{j=1}^{i} (R_j+1)] Y_{i:m:n}^{2\alpha}}\right]. \end{aligned} \tag{18}$$

We notice that $W(2\alpha)$ is a function of α and does not depend on σ. Moreover, $W(2\alpha) = \sum_{i=1}^{m-1}(-2logU_{(i)}) = \sum_{i=1}^{m-1}(-2log(U_i))$, and $U_1, ..., U_{m-1} = \frac{S_1}{S_m}, ..., \frac{S_{m-1}}{S_m}$ is a random sample from the uniform(0,1) distribution which implies that $W(2\alpha)$ has the χ^2 distribution with $2(m-1)$ degrees of freedom. To show that $W(2\alpha)$ is strictly monotonic function, we write equation 18 as follows.

$$\begin{aligned} W(2\alpha) &= 2\sum_{i=1}^{m-1} log\left(1 + \frac{S_m - S_i}{S_i}\right) \\ &= 2\sum_{i=1}^{m-1} log\left(1 + \frac{\sum_{j=i+1}^{m}(R_j+1)V_{j:m:n} - [n - \sum_{j=1}^{i}(R_j+1)]V_{i:m:n}}{\sum_{j=1}^{i}(R_j+1)V_{j:m:n} + [n - \sum_{j=1}^{i}(R_j+1)]V_{i:m:n}}\right) \\ &= 2\sum_{i=1}^{m-1} log\left(1 + \frac{\sum_{j=i+1}^{m}(R_j+1)P_{(j,i)} - [n - \sum_{j=1}^{i}(R_j+1)]}{\sum_{j=1}^{i}(R_j+1)P_{(j,i)} + n - \sum_{j=1}^{i}(R_j+1)}\right), \end{aligned} \quad (19)$$

where $P_{(j,i)} = \frac{V_{j:m:n}}{V_{i:m:n}} = \left(\frac{Y_{j:m:n}}{Y_{i:m:n}}\right)^{2\alpha}$ is strictly increasing in α for $j > i$, and then $W(2\alpha)$ is strictly increasing function of α. Therefore, W^{-1} exists and the confidence interval of the parameter α is stated in the following theorem.

Theorem 2. *Suppose $X = (X_{1:m:n}, ..., X_{m:m:n})$ is a progressively Type II right censored sample from the RL distribution with sample of size n and the censoring scheme $R = (R_1, ..., R_m)$. Then, for any $0 < \gamma < 1$,*

$$\left[\tfrac{1}{2}W^{-1}[\chi^2_{1-\gamma/2}(2(m-1))], \tfrac{1}{2}W^{-1}[\chi^2_{\gamma/2}(2(m-1))]\right]$$

is a $100(1-\gamma)\%$ confidence interval for the shape parameter α, where $\chi^2_{1-\gamma/2}(2(m-1))$ and $\chi^2_{\gamma/2}(2(m-1))$ are the lower and upper γ percentiles respectively of the χ^2 distribution with $2(m-1)$ degrees of freedom.

3.2 Interval estimation of parameter σ

To obtain the confidence interval of σ, we consider the quantity, $V = 2S_m$. Note that from part (III), $S_m = \sum_{j=1}^{m} W_j$. Hence,

$$\begin{aligned} V &= 2\sum_{j=1}^{m} W_j \\ &= 2\sum_{j=1}^{m}(R_j+1)V_{j:m:n} \end{aligned} \quad (20)$$

For the Trans-RL distribution, the quantity V can be written as

$$V = \frac{1}{\sigma^2}\sum_{j=1}^{m}(R_j+1)Y_{j:m:n}^{2g(W,Y)}, \quad (21)$$

where $g(W,Y) = \alpha = \frac{1}{2}W^{-1}(t)$ obtained from (18) numerically and V has the χ^2 distribution with $2m$ degrees of freedom. Hence, the $100(1-\gamma)\%$ confidence interval of σ is

$$\left[\sqrt{\frac{\sum_{j=1}^{m}(R_j+1)Y_{j:m:n}^{2g(W,Y)}}{\chi^2_{\gamma/2}(2m)}}, \sqrt{\frac{\sum_{j=1}^{m}(R_j+1)Y_{j:m:n}^{2g(W,Y)}}{\chi^2_{1-\gamma/2}(2m)}}\right]$$

.

3.3 Inverse estimation of parameters α and σ

Since $W(2\alpha)$ has the χ^2 distribution with $2(m-1)$ degrees of freedom and $E(W(2\alpha)) = 2(m-1) < \infty$, then by strong law of large numbers, $W(2\hat{\alpha}) \overset{a.s}{\to} 2(m-2)$ (or $W(2\hat{\alpha})$ converges with probability one to $2(m-2)$). Therefore, we can obtain the point estimator $\hat{\alpha}$ of α from the following equation:

$$W(2\hat{\alpha}) = 2(m-2). \tag{22}$$

The inverse estimate of α is obtained by solving equation (22) numerically. From the previous subsection, we know that $V = 2S_m$ has the χ^2 distribution with $2m$ degrees of freedom. Hence, the inverse estimate of the parameter σ is

$$\hat{\sigma} = \sqrt{\frac{\sum_{j=1}^{m}(R_j+1)Y_{j:m:n}^{2\hat{\alpha}}}{2(m-1)}}. \tag{23}$$

4 Simulation Results And Illustrative Example

4.1 Simulation Results

In this subsection, we conduct a simulation study for the Trans-RL distribution under a variety of progressively Type-II right censored sampling schemes over 10000 replications.

We generate progressively Type-II censored samples from the Trans-RL distribution for different choices of sample sizes and censoring schemes provided by Wang et al. (2010). Table 1 shows the coverage probabilities of confidence intervals of α and σ at 0.90 and 0.95 confidence levels for the Trans-RL distribution. It illustrates that the simulated probabilities for 0.90 and 0.95 are very close to the 0.90 and 0.95 confidence levels. We also obtain the inverse estimate of α and σ in Table 2 and Table 3 and compare their performance with the maximum likelihood estimates (MLEs) and L-moment estimates which are presented in Table 4-6, where L-moments can be estimated by linear combinations of order statistics (Hosking (1990)). We observe that the inverse estimation provides a good alternative to the method of L-moments and MLE in terms of bias and

MSE. Moreover, in almost all cases, the estimators of the parameters α and σ become less biased as the censored units increase for a fixed sample size.

Table 1: The coverage probabilities of the confidence intervals in the transformed RL distribution for $\alpha = 1$ and $\sigma = 1$.

		α		σ	
(n,m)	(r1,...,rm)	0.90	0.95	0.90	0.95
(10,5)	(0,...,0,5)	0.8500	0.9149	0.8416	0.9100
(10,5)	(5,0,...,0)	0.8771	0.9368	0.8591	0.9223
(10,8)	(0,0,...,2)	0.8288	0.8975	0.8781	0.9346
(10,8)	(2,0,...,0)	0.8447	0.9102	0.8722	0.9334
(20,10)	(0,0,...,10)	0.8401	0.9050	0.8755	0.9348
(20,10)	(10,0,...,0)	0.8557	0.9190	0.8776	0.9364
(20,15)	(0,0,...,5)	0.8385	0.9025	0.8877	0.9415
(20,15)	(5,0,...,0)	0.8469	0.9125	0.8838	0.9403
(30,10)	(0,0,...,20)	0.8520	0.9146	0.8790	0.9370
(30,10)	(20,0,...,0)	0.8626	0.9234	0.8836	0.9393
(50,12)	(0,...,0,38)	0.8694	0.9298	0.8881	0.9406
(50,12)	(38,0,...,0)	0.8725	0.9293	0.8820	0.9366
(50,25)	(0,0,...,25)	0.8457	0.9102	0.8872	0.9396
(50,25)	(25,0,...,0)	0.8410	0.9056	0.8948	0.9446

Table 2: The average bias and average MSE of the inverse estimators of the parameters of the transformed RL distribution for $\alpha = 1$ and $\sigma = 1$.

		Bias		MSE	
(n,m)	(r1,...,rm)	$\hat{\alpha}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\sigma}$
(10,5)	(0,...,0,5)	-0.04258	0.27993	0.32956	0.75423
(10,5)	(5,0,...,0)	-0.03892	0.19442	0.162947	0.59069
(10,8)	(0,...,0,2)	-0.00334	0.20340	0.10729	0.31404
(10,8)	(2,0,...,0)	-0.02781	0.14724	0.10363	0.50432
(20,10)	(0,0,...,10)	-0.01636	0.10813	0.11379	0.18029
(20,10)	(10,0,...,0)	-0.03189	0.07033	0.05969	0.19855
(20,15)	(0,0,...,5)	0.00031	0.09648	0.07304	0.22366
(20,15)	(5,0,...,0)	-0.01603	0.08205	0.04257	0.15048
(30,10)	(0,...,0,20)	-0.00912	0.13493	0.12528	0.16423
(30,10)	(20,0,...,0)	-0.01380	0.06646	0.05292	0.16163
(50,12)	(0,...,0,38)	0.00532	0.11155	0.10696	0.14701
(50,12)	(38,0,...,0)	-0.00705	0.06631	0.04168	0.14275
(50,25)	(0,...,0,25)	-0.01335	0.04005	0.03567	0.04766
(50,25)	(25,0,...,0)	-0.00398	0.02737	0.02273	0.07204

Table 3: The inverse estimates of the parameters of the transformed RL distribution for $\alpha = 1$ and $\sigma = 1$.

(n,m)	(r1,...,rm)	$\hat{\alpha}$	$\hat{\sigma}$
(10,5)	(0,0...,5)	0.95742	1.27993
(10,5)	(5,0,...,0)	0.96108	1.19442
(10,8)	(0,...,0,2)	0.99666	1.20340
(10,8)	(2,0,...,0)	0.97219	1.14724
(20,10)	(0,0,...,10)	0.98364	1.10813
(20,10)	(10,0,...,0)	0.96811	1.07033
(20,15)	(0,...,0,5)	1.00031	1.09648
(20,15)	(5,0,...,0)	0.98397	1.08205
(30,10)	(0,...,0,20)	0.99088	1.13493
(30,10)	(20,0,...,0)	0.98620	1.06646
(50,12)	(0,...,0,38)	1.00532	1.11155
(50,12)	(38,0,...,0)	0.99295	1.06631
(50,25)	(0,...,0,25)	0.98665	1.04005
(50,25)	(25,0,...,0)	0.99602	1.04342

Table 4: The average bias of the L-moments and MLEs of the parameters of the Transformed RL distribution

		Biase			
		$\hat{\alpha}$		$\hat{\sigma}$	
(n,m)	(r1,...,rm)	L-mom	MLE	L-mom	MLE
(10,5)	(0,..,0,5)	0.55567	0.92258	-0.38858	-0.79442
(10,5)	(5,0,...,0)	-0.07781	0.15797	-0.07739	-0.10608
(10,5)	(1,1,...,1)	0.15453	0.42711	-0.26749	-0.44135
(10,8)	(0,...,0,2)	0.27196	0.45911	-0.11571	-0.3555
(10,8)	(2,0,...,0)	0.01472	0.16016	0.00327	0.01753
(20,10)	(0,...,0,10)	0.42648	0.61213	-0.42983	-0.61913
(20,10)	(10,0,...,0)	-0.05016	0.05026	-0.05732	-0.08407
(20,10)	(1,1,...,1)	0.03664	0.14912	-0.28479	-0.48656
(20,15)	(0,...,0,5)	0.28703	0.39965	-0.18612	-0.25265
(20,15)	(5,0,...,0)	0.00450	0.07531	-0.00134	0.01254
(30,10)	(0,...,0,20)	0.50733	0.70672	-0.60172	-0.64089
(30,10)	(20,0,...,0)	-0.10060	-0.01042	-0.08483	-0.11747
(30,10)	(2,2,...,2)	0.07229	0.18551	-0.43015	-0.51704
(30,20)	(0,...,0,10)	0.31000	0.41319	-0.28147	-0.31776
(50,12)	(38,0,...,0)	-0.11779	-0.05355	-0.09259	-0.10947
(50,25)	(25,0,...,0)	-0.03584	-0.00130	-0.02766	-0.01838
(50,25)	(1,1,...,1)	0.02812	0.06811	-0.28931	-0.30948

Table 5: The average MSE of the L-moments and MLEs of the parameters of the Transformed RL distribution

		MSE			
		$\hat{\alpha}$		$\hat{\sigma}$	
(n,m)	(r1,...,rm)	L-mom	MLE	L-mom	MLE
(10,5)	(0,..,0,5)	1.50534	2.34460	0.31145	1.13771
(10,5)	(5,0,...,0)	0.16243	0.25583	0.07078	0.20065
(10,8)	(0,...,0,2)	0.29683	0.51212	0.10391	0.70489
(10,8)	(2,0,...,0)	0.12723	0.19352	0.08648	0.23226
(20,10)	(0,...,0,10)	0.42118	0.68416	0.20333	0.56756
(20,10)	(10,0,...,0)	0.06158	0.07473	0.04174	0.16764
(20,15)	(0,...,0,5)	0.17883	0.28198	0.06654	0.22357
(20,15)	(5,0,...,0)	0.04760	0.05868	0.02974	0.06064
(30,10)	(0,...,0,20)	0.50528	0.83076	0.37277	0.42960
(30,10)	(20,0,...,0)	0.05666	0.05525	0.03644	0.15508
(30,20)	(0,...,0,10)	0.17262	0.26647	0.09526	0.16951
(50,12)	(38,0,...,0)	0.04869	0.04024	0.03231	0.09948
(50,25)	(25,0,...,0)	0.02286	0.02195	0.02030	0.01638
(50,25)	(1,1,...,1)	0.03492	0.03903	0.09031	0.13263

Table 6: The maximum likelihood and L-moment estimates of the parameters of the Transformed RL distribution

		Estimate			
		$\hat{\alpha}$		$\hat{\sigma}$	
(n,m)	(r1,...,rm)	L-mom	MLE	L-mom	MLE
(10,5)	(0,...,0,5)	1.47614	1.97562	0.61142	0.20558
(10,5)	(5,0,...,0)	0.92219	1.15797	0.92261	0.89392
(10,5)	(1,1,...,1)	1.15454	1.42711	0.73251	0.55867
(10,8)	(0,0,...,2)	1.27196	1.45911	0.88428	0.64447
(10,8)	(2,0,...,0)	1.01472	1.16015	1.00327	1.01753
(20,10)	(0,0,...,10)	1.42648	1.61213	0.57017	0.38087
(20,10)	(10,0,...,0)	0.94983	1.05026	0.94267	0.91593
(20,10)	(1,1,...,1)	1.03664	1.14912	0.71520	0.51344
(20,15)	(0,0,...,5)	1.28703	1.39965	0.81388	0.74734
(20,15)	(5,0,...,0)	1.00450	1.07531	0.99866	1.01254
(30,10)	(0,0,...,20)	1.50733	1.70672	0.39828	0.35910
(30,10)	(20,0,...,0)	0.89939	0.98958	0.91517	0.88253
(30,10)	(2,2,...,2)	1.07229	1.18551	0.56985	0.48296
(30,20)	(0,0,...,10)	1.31000	1.41319	0.71853	0.68223
(50,12)	(38,0,...,0)	0.88221	0.94645	0.90741	0.89053
(50,25)	(25,0,...,0)	0.96416	0.99869	0.97234	0.98162
(50,25)	(1,1,...,1)	1.02812	1.06811	0.71069	0.69052

4.2 An Illustrative Example

We consider the following general progressively Type-II censored data which represent the time (in minutes) to breakdown of an insulating fluid between electrodes at voltage 30 kv. This data is taken from Nelson (1982, Table 6.1, p. 228). The complete data set consist of $n = 11$ times to breakdown. The progressively censored data are given as follows

r_i	0	0	0	0	3	0	0	0
Y_i	2.0464	2.8361	3.0184	3.0454	3.1206	4.9706	5.1698	5.2724

The experimenter removed three survival units from the test at the failure (breakdown) of an insulating fluid which is occurred at 3.1206 minutes such that $\sum_{i=1}^{8} r_i + m = 3 + 8 = 11$.

The maximum likelihood and inverse estimates of α and σ are computed and the results are shown in Table 7.

Table 7: The maximum likelihood and inverse estimates of α and σ

	α		σ	
	MLE	Inverse	MLE	Inverse
Complete data	1.84201	1.63638	9.73321	7.45820
Progressive data	1.74957	1.67150	8.38262	8.53374

From Table 7, we observe that the inverse estimates of the Trans-RL parameters α and σ in the case of progressively censored sample are closer to the ones based on the complete sample (when $m = n$ and $R_1 = ... = R_m = 0$) than the MLEs. Hence, the inverse estimation is preferable and is considered as a good alternative even for small sample size. Moreover, the confidence intervals at 0.90 and 0.95 confidence levels for each of α and σ are calculated and shown in Table 8 and Table 9.

Table 8: The 0.90 confidence intervals for the parameters α and σ.

	90% confidence limits			
	α		σ	
	Lower	Upper	Lower	Upper
Complete data	1.07606	2.59416	6.27758	10.68066
Progressive data	1.05046	2.77078	6.56098	12.45661
Complete data (MLE)	1.07657	2.60781	-1.97060	21.44285
Progressive data (MLE)	1.02828	2.47307	-1.18758	17.98259

Table 9: The 0.95 confidence intervals for the parameters α and σ.

	95% confidence limits			
	α		σ	
	Lower	Upper	Lower	Upper
Complete data	0.97022	2.78400	6.01879	11.36063
Progressive data	0.92780	2.98112	6.24778	13.45855
Complete data (MLE)	1.06161	2.62278	-2.19938	21.67163
Progressive data (MLE)	1.04213	2.45922	-1.00412	17.79886

Tables 8 and 9 show that the confidence intervals for both α and σ derived in sections 3.1 and 3.2 are shorter than the confidence intervals based on the maximum likelihood estimation in almost all cases, which indicates that the confidence intervals using Wang et al. (2010) method outperform those of maximum likelihood method.

5 Discussion

In this paper, we introduce an inference under progressively Type-II censoring for a new generalization of Lomax distribution after transformation, called Trans-RL distribution. This distribution provides great fit in modeling wide range of real datasets. According to Wang et al. (2010), we discuss some properties of order statistics and Poisson processes as the Trans-RL distribution belongs to proportional hazard family. Therefore, we derive the confidence intervals and inverse estimators of the proposed distribution. Simulation study is performed to investigate the coverage probabilities, the average biases and the average relative mean square errors for the maximum likelihood, the method of L-moments and the inverse estimators. Therefore, we show that the performance of the inverse estimation and the confidence intervals proposed in this paper perform quite better than the ones derived from the maximum likelihood and the L-moments estimators under different sample sizes and censoring schemes. Finally, a numerical example is provided to explain the purpose of this study.

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