2007

Unified Analysis of Finite Volume Methods for Second Order Elliptic Problems

So-Hsiang Chou
Bowling Green State University, chou@bgsu.edu

Xiu Ye

SIAM J. NUMER. ANAL.
Vol. 45, No. 4, pp. 1639–1653

UNIFIED ANALYSIS OF FINITE VOLUME METHODS FOR SECOND ORDER ELLIPTIC PROBLEMS*

SO-HSIANG CHOU† AND XIU YE‡

Abstract. We establish a general framework for analyzing the class of finite volume methods which employ continuous or totally discontinuous trial functions and piecewise constant test functions. Under the framework, optimal order convergence in the H^1 and L^2 norms can be obtained in a natural and systematic way for classical finite volume methods and new finite volume methods such as discontinuous finite volume methods applied to second order elliptic problems.

Key words. finite element methods, finite volume methods, discontinuous Galerkin methods, finite volume element

AMS subject classifications. Primary, 65N15, 65N30, 76D07; Secondary, 35B45, 35J50

DOI. 10.1137/050643994

1. Introduction.

Due to the local conservation property and other attractive properties such as robustness with unstructured meshes, the finite volume method is widely used in computational fluid dynamics. Numerical analysis of a finite volume method is more difficult than that of a finite element method, since in general a finite volume method uses two different function spaces: one for the trial space and one for the test space. For example, obtaining the optimal L^2 error estimates is a common practice for finite element methods. They are very difficult to obtain for the finite volume methods. Because of this reason, the optimal L^2 estimates have not been derived for the finite volume methods proposed in [8, 9, 10, 13, 25]. The main motivation of this paper is to propose a general framework under which we can systematically give a thorough analysis for finite volume methods to second order elliptic problems and obtain the optimal error estimates in energy norm and L^2 norm.

In recent years, there have appeared different approaches in the convergence and stability analysis of the finite volume method; see, for example, [2, 5, 6, 12, 13, 16, 15, 17, 18, 22], among others. Motivated by the popularity of discontinuous Galerkin methods, Ye [25] proposed a finite volume method with a totally discontinuous trial function space for elliptic problems. Our general framework covers the finite volume methods (continuous or discontinuous) developed in all of the papers mentioned above in a unified way, and previously hard-to-obtain optimal L^2 estimates [8, 10, 9, 13, 25] can now be derived naturally.

For simplicity in this paper we will treat only finite volume methods applied to the self-adjoint elliptic equations. To illustrate the idea, we consider the model problem

$$\mathcal{L}u := -\nabla \cdot \mathcal{A}\nabla u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where $\Omega \subset R^2$ is a bounded polygonal domain and $\mathcal{A}$ is in either $W^{1,\infty}$ or $W^{2,\infty}$. A typical finite volume method uses piecewise constant functions as test functions, and,

*Received by the editors November 1, 2005; accepted for publication (in revised form) May 8, 2007; published electronically August 17, 2007.
http://www.siam.org/journals/sinum/45-4/64399.html

†Department of Mathematics and Statistics, Bowling Green State University, Bowling Green, OH 43403-0221 (chou@bgnet.bgsu.edu). The work of this author was supported in part by the National Center for Theoretical Sciences, Taiwan.

‡Department of Mathematics, University of Arkansas at Little Rock, Little Rock, AK 72204 (xxye@ualr.edu).

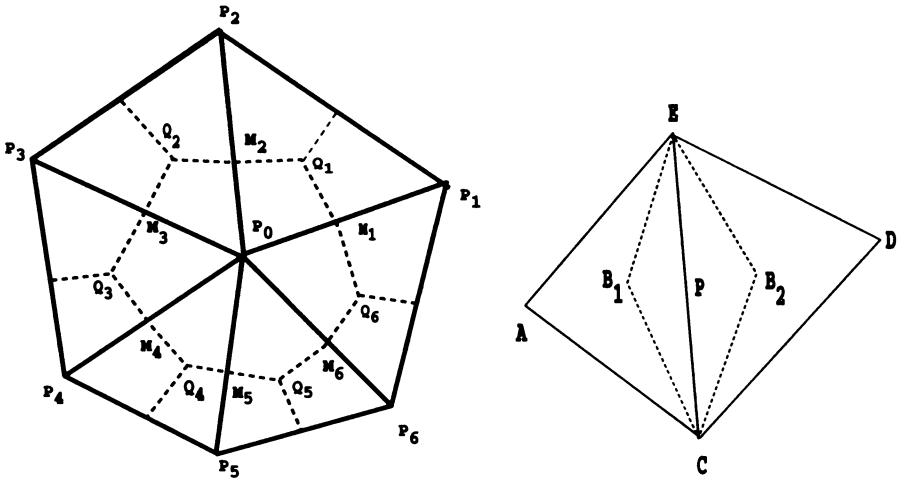

FIG. 1. *Primal and dual grids. Left figure: Conforming finite volume method. Right figure: Nonconforming finite volume method.*

to keep the same dimension for the spaces of the trial functions and test functions, two different partitions of the domain Ω are needed: one called the primal partition is associated with the trial space, and one called the dual partition is associated with the test space. For example, in Figure 1, on the left the primal partition is made up of the standard triangular finite elements, and the dual partition is the usual barycentric subdivision consisting of polygons around P_i's obtained by connecting midpoints M_i's of edges and barycenters Q_i's of the triangles. Thus $M_1Q_1M_2Q_2M_3Q_3M_4Q_4M_5Q_5M_6Q_6$ is a typical dual volume around P_0. On the other hand, in the right figure of Figure 1 we use triangles in the primal partition, and for each midpoint of an edge in the triangles we define a quadrilateral element that serves as an element in the dual partition. So, for example, in Figure 1 the quadrilateral EB_1CB_2 around midpoint P (B_i barycenters of triangles) is in the dual partition.

Figure 2 shows two more possible configurations of primal (solid lines) and dual (dashed lines) partitions. In particular, the partitions in the right figure will be used for the discontinuous finite volume method in section 3.3. Here we use standard triangular elements in the primal partition, and each triangular element then generates three dual triangular volumes (AB_1D and two others) by connecting its barycenter and vertices.

Denote by $\mathcal{T}_h$ the primal triangulation of Ω, by $\mathcal{T}_h^*$ the dual partition of $\mathcal{T}_h$, and by $P_l(T)$ the space of all polynomials on T whose degree is at most l. The finite dimensional trial space V_h associated with $\mathcal{T}_h$ is a subspace of piecewise linears, i.e.,

$$V_h \subset \{v \in V : \ v|_T \in P_1(T) \ \forall T \in \mathcal{T}_h\}, \tag{1.2}$$

where V is either $H_0^1(\Omega)$ or $L^2(\Omega)$ (standard Sobolev spaces notation will be adopted throughout the paper). Examples of such space are continuous P_1 conforming space, the Crouzeix–Raviart P_1 nonconforming space [14] (continuous at midpoints), and totally discontinuous P_1 space to be used in conjunction with the discontinuous finite volume method in section 3.3. The test function space Q_h associated with the dual partition $\mathcal{T}_h^*$ is

$$Q_h = \{q \in L^2(\Omega) : \ q|_K \in P_0(K) \ \ \forall K \in \mathcal{T}_h^*\}. \tag{1.3}$$

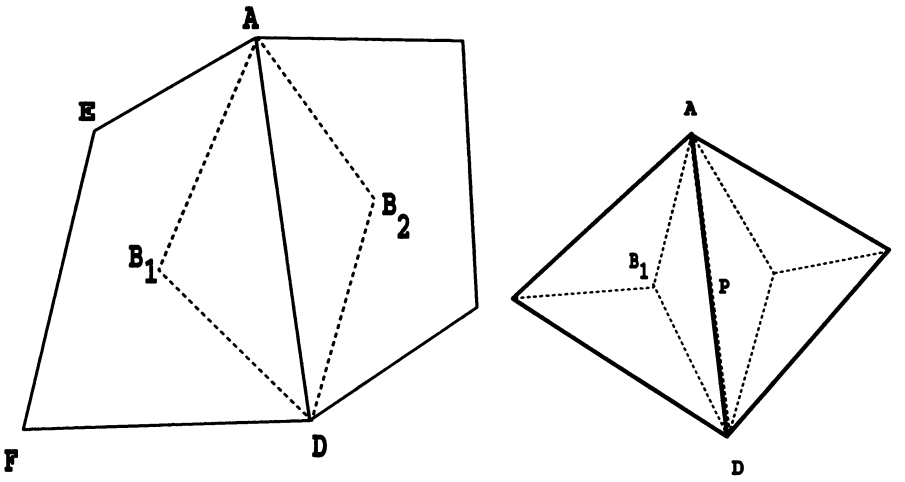

FIG. 2. *Primal and dual grids. Left figure: Nonconforming finite volume method. Right figure: Discontinuous finite volume method.*

We mention in passing that classical finite volume methods adopt piecewise P_0 shape functions, and their applications abound. The present (and newer) finite volume methods using piecewise P_1 shape functions also find many practical applications in heat transfer and fluid flow problems [7, 21] and the references therein. These methods are also natural when combined with the multilevel adaptive methods [19, 20].

Due to the efforts of several authors [6, 12, 15, 17], especially [6, 15, 17], it is now recognized that, for finite volume methods applied to second order elliptic problems on polygonal domains, it is to be expected that, for the exact solution u and approximate solution u_h, the best form of the L^2 estimates is

$$||u - u_h|| \leq Ch^2(||u||_2 + ||f||_1).$$

(We use $||\cdot||_p$ for the standard Sobolev H^p norm and drop the subindex for the L^2 norm.) One notes that this is not the same as assuming u in $H^3(\Omega)$. For example, the solution of the boundary value problem $\Delta u = 1$ on the unit square and $u = 0$ on the boundary belongs to $H^2(\Omega)$ but not to $H^3(\Omega)$. While it is easy and natural to deduce the above error estimates under our present framework, it should be pointed out that there are other ways to view finite volume methods, depending on how one views what the distinctive traits of a finite volume method are. For example, one may consider the so-called mixed finite volume method in which the flux can be recovered by a simple formula [11]. On the other hand, in other finite volume methods the flux itself plays an important role in the derivation of the method. For instance, in [16], finite volume methods are based on considering averages of solutions on the control volumes which coincide with the supports of the test functions in the present paper. The stiffness matrix is calculated from a difference approximation of the fluxes between two neighboring elements. Compactness methods are used to prove the convergence. While this approach can be generalized consistently to convection-diffusion and hyperbolic problems, it shows considerable difficulties when error estimates are to be obtained. Our approach focuses on a narrower elliptic problem class and explores its natural relation to the Galerkin finite element method. Consequently, optimal order error estimates are easier to obtain.

The organization of the paper is as follows. In section 2 we present our general finite volume framework and its stability and convergent analysis. Under this framework, in section 3 we systematically derive for the new as well as the old finite volume methods the optimal H^1 estimates of the usual form and optimal L^2 estimates of the above form.

Let e be an interior edge common to elements T_1 and T_2 in $\mathcal{T}_h$, and let $\mathbf{n}_1$ and $\mathbf{n}_2$ be the unit normal vectors on e exterior to K_1 and K_2, respectively. For a scalar q and a vector $\mathbf{w}$ we define their average $\{\cdot\}$ on e and jump $[\![\cdot]\!]$ across e, respectively, as

$$\{q\} = \frac{1}{2}(q|_{\partial T_1} + q|_{\partial T_2}), \quad [\![q]\!] = q|_{\partial T_1}\mathbf{n}_1 + q|_{\partial T_2}\mathbf{n}_2,$$

$$\{\mathbf{w}\} = \frac{1}{2}(\mathbf{w}|_{\partial T_1} + \mathbf{w}|_{\partial T_2}), \quad [\![\mathbf{w}]\!] = \mathbf{w}|_{\partial T_1} \cdot \mathbf{n}_1 + \mathbf{w}|_{\partial T_2} \cdot \mathbf{n}_2.$$

Note that the jump of a vector is a scalar, whereas the jump of a scalar is a vector. If e is an edge on the boundary of Ω, we define

$$\{q\} = q, \quad [\![\mathbf{w}]\!] = \mathbf{w} \cdot \mathbf{n}.$$

The quantities $[\![q]\!]$ and $\{\mathbf{w}\}$ on boundary edges are defined analogously. Let $\mathcal{E}_h$ denote the union of the boundaries of the triangles T of $\mathcal{T}_h$ and $\mathcal{E}_h^0 := \mathcal{E}_h \backslash \partial\Omega$ the collection of all interior edges.

Following [8, 12], we assume the existence of a transfer operator γ from $V(h) := V_h + H^2(\Omega) \cap H_0^1(\Omega)$ to the test space Q_h. In particular, γ connects the trial space V_h with the test space Q_h. Throughout the paper, the operator γ is required to satisfy the following sets of assumptions.

Assumption 1. Quadraturelike and restriction assumptions for γ:

$$\int_T (v - \gamma v)dx = 0 \quad \forall v \in V_h, \ \forall T \in \mathcal{T}_h, \tag{1.4}$$

$$\int_e (v - \gamma v)ds = 0 \quad \forall v \in H^2(\mathcal{T}_h), \ \forall e \in \partial T, \ \forall T \in \mathcal{T}_h, \tag{1.5}$$

$$\text{if} \quad [\![v]\!] = 0, \quad \text{then} \quad [\![\gamma v]\!] = 0, \tag{1.6}$$

where $H^2(\mathcal{T}_h) := \{v \in L^2(\Omega) : v|_T \in H^2(T) \qquad \forall T \in \mathcal{T}_h\}$.

Equations (1.4)–(1.5) have been observed in [12, 13] and perhaps can be viewed as a type of quadrature condition. Equation (1.6) is our new observation in this paper regarding to the jump.

Assumption 2. Approximation property of γ:

$$||\gamma w - w||_{0,T} \leq Ch_T|w|_{1,T} \qquad \forall T \in \mathcal{T}_h. \tag{1.7}$$

Then the solution of (1.1) necessarily satisfies

$$\mathcal{L}u = -\nabla \cdot \mathcal{A}\nabla u = f \quad \text{on } K \quad \forall K \in \mathcal{T}_h^*, \tag{1.8}$$

$$[\![\gamma u]\!]_e = 0 \quad \forall e \in \mathcal{E}_h, \tag{1.9}$$

$$[\![\mathcal{A}\nabla u]\!]_e = 0 \quad \forall e \in \mathcal{E}_h^0. \tag{1.10}$$

2. Finite volume formulation. In this section, we will derive a general formulation for finite volume methods. The formulation is based on enforcing (1.8)–(1.10) by testing with "element" test functions for (1.8) and "edge" test functions for (1.9)

and (1.10). To this end, we further assume the existence of two linear operators $B_1 : V(h) \to L^2(\mathcal{E}_h)$ and $B_2 : V(h) \to L^2(\mathcal{E}_h^0)$ (they will be defined shortly). Testing (1.8), (1.9), and (1.10) by γv, $B_1 v$, and $B_2 v$, respectively, and adding them up, we obtain the "global" equation

$$(\mathcal{L}u, \gamma v)_{\mathcal{T}_h^*} + (\llbracket \gamma u \rrbracket, B_1 v)_{\mathcal{E}_h} + (\llbracket \mathcal{A}\nabla u \rrbracket, B_2 v)_{\mathcal{E}_h^0} = (f, \gamma v), \tag{2.1}$$

where each inner product obviously means the sum of its local inner products. *A remark is in order here.* Interpreting PDEs and jump conditions such as (1.8)–(1.10) as residual equations and testing them with test functions of different levels is, of course, quite common in finite element and finite volume methods. However, the fact that summing them up as equal weight relations can lead to fruitful analysis is more recent. In fact, using this technique Brezzi et al. [4] have demonstrated stabilization mechanisms in discontinuous Galerkin methods in a unified way.

Integrating (2.1) by parts and using the fact that γv is constant on K, we have

$$\begin{aligned}
(\mathcal{L}u, \gamma v)_{\mathcal{T}_h^*} &= -\sum_{K \in \mathcal{T}_h^*} \int_K \nabla \cdot \mathcal{A}\nabla u \gamma v dx \\
&= -\sum_{K \in \mathcal{T}_h^*} \int_{\partial K} \mathcal{A}\nabla u \cdot \mathbf{n} \gamma v ds \\
&= \left(-\sum_{K \in \mathcal{T}_h^*} \int_{\partial K} \mathcal{A}\nabla u \cdot \mathbf{n} \gamma v ds + \sum_{T \in \mathcal{T}_h} \int_{\partial T} \mathcal{A}\nabla u \cdot \mathbf{n} \gamma v ds \right) \\
&\quad - \sum_{T \in \mathcal{T}_h} \int_{\partial T} \mathcal{A}\nabla u \cdot \mathbf{n} \gamma v ds,
\end{aligned}$$

where we have added and subtracted the last term to bring in the effect of primal triangulation.

Define the bilinear form $a : V(h) \times V(h) \to R$

$$a(u, v) := -\sum_{K \in \mathcal{T}_h^*} \int_{\partial K} \mathcal{A}\nabla u \cdot \mathbf{n} \gamma v ds + \sum_{T \in \mathcal{T}_h} \int_{\partial T} \mathcal{A}\nabla u \cdot \mathbf{n} \gamma v ds.$$

Recall the following easily derived identity (or see [1]): For all $q \in \prod_{T \in \mathcal{T}_h} L^2(\partial T)$ and for all $\mathbf{v} \in [\prod_{T \in \mathcal{T}_h} L^2(\partial T)]^2$,

$$\sum_{T \in \mathcal{T}_h} \int_{\partial T} q \mathbf{v} \cdot \mathbf{n} ds = \int_{\mathcal{E}_h} \llbracket q \rrbracket \{\mathbf{v}\} ds + \int_{\mathcal{E}_h^0} \{q\} \llbracket \mathbf{v} \rrbracket ds. \tag{2.2}$$

In particular,

$$\sum_{T \in \mathcal{T}_h} \int_{\partial T} \mathcal{A}\nabla u \cdot \mathbf{n} \gamma v ds = \sum_{e \in \mathcal{E}_h} \int_e \llbracket \gamma v \rrbracket \cdot \{\mathcal{A}\nabla u\} ds + \sum_{e \in \mathcal{E}_h^0} \int_e \{\gamma v\} \llbracket \mathcal{A}\nabla u \rrbracket ds, \tag{2.3}$$

and hence (2.1) becomes

$$\begin{aligned}
a(u, v) &- (\llbracket \mathcal{A}\nabla u \rrbracket, \{\gamma v\})_{\mathcal{E}_h^0} - (\{\mathcal{A}\nabla u\}, \llbracket \gamma v \rrbracket)_{\mathcal{E}_h} \\
&+ (\llbracket \gamma u \rrbracket, B_1 v)_{\mathcal{E}_h} + (\llbracket \mathcal{A}\nabla u \rrbracket, B_2 v)_{\mathcal{E}_h^0} = (f, \gamma v).
\end{aligned}$$

The choice of $B_2 v = \{\gamma v\}$ leads to

$$a(u,v) - (\{\mathcal{A}\nabla u\}, [\![\gamma v]\!])_{\mathcal{E}_h} + ([\![\gamma u]\!], B_1 v)_{\mathcal{E}_h} = (f, \gamma v).$$

Furthermore, if we take the common pick of $B_1 v = \alpha h^{-1}[\![\gamma v]\!] + \delta\{\mathcal{A}\nabla v\}$, where α is a positive number and $\delta = 1, -1$, the above equation becomes

$$a(u,v) - (\{\mathcal{A}\nabla u\}, [\![\gamma v]\!])_{\mathcal{E}_h} + \delta([\![\gamma u]\!], \{\mathcal{A}\nabla v\})_{\mathcal{E}_h} + \alpha h^{-1}([\![\gamma u]\!], [\![\gamma v]\!])_{\mathcal{E}_h} = (f, \gamma v).$$

For simplicity, we will fix our choices and take $B_1 v = \alpha h^{-1}[\![\gamma v]\!] + \delta\{\mathcal{A}\nabla v\}$ and $B_2 v = \{\gamma v\}$ in the remaining part of the paper. However, our analysis carries through for other choices in [4] as well.

Let

$$A(u,v) := a(u,v) - (\{\mathcal{A}\nabla u\}, [\![\gamma v]\!])_{\mathcal{E}_h} + \delta([\![\gamma u]\!], \{\mathcal{A}\nabla v\})_{\mathcal{E}_h} + \alpha h^{-1}([\![\gamma u]\!], [\![\gamma v]\!])_{\mathcal{E}_h}, \tag{2.4}$$

and consider the following class of finite volume methods: Find $u_h \in V_h$

$$A(u_h, v) = (f, \gamma v) \qquad \forall v \in V_h. \tag{2.5}$$

The formulation (2.5) is consistent; i.e., the true solution u satisfies

$$A(u,v) = (f, \gamma v) \quad \forall v \in V_h. \tag{2.6}$$

Subtracting (2.5) from (2.6) gives

$$A(u - u_h, v) = 0 \quad \forall v \in V_h. \tag{2.7}$$

We define a norm $|\!|\!| \cdot |\!|\!|$ on $V(h)$ as

$$|\!|\!| v |\!|\!|^2 = |u|_{1,h}^2 + \sum_{e \in \mathcal{E}_h} [\![\gamma v]\!]_e^2 + \sum_{T \in \mathcal{T}_h} h_T^2 |v|_{2,T}^2.$$

We assume the bilinear for $A(\cdot,\cdot)$ is bounded and coercive:

Assumption 3.

$$|A(v,w)| \le C_1 |\!|\!| v |\!|\!| |\!|\!| w |\!|\!| \qquad \forall v, w \in V(h) \times V(h), \tag{2.8}$$

$$A(v,v) \ge C_2 |\!|\!| v |\!|\!|^2 \qquad \forall v \in V_h. \tag{2.9}$$

Then we have the following theorem that is the counterpart of Céa's lemma [3] in the finite element theory.

THEOREM 2.1. *Let u and u_h be the solutions of* (1.1) *and* (2.5). *Then*

$$|\!|\!| u - u_h |\!|\!| \le C \inf_{v \in V_h} |\!|\!| u - v |\!|\!|.$$

Proof. From (2.9) and (2.7), we have that for any $v \in V_h$

$$C_1 |\!|\!| u_h - v |\!|\!|^2 \le A(u_h - v, u_h - v) = A(u - v, u_h - v) \le C_2 |\!|\!| u - v |\!|\!| |\!|\!| u_h - v |\!|\!|.$$

Hence by the triangle inequality we have

$$|\!|\!| u - u_h |\!|\!| \le C \inf_{v \in V_h} |\!|\!| u - v |\!|\!|.$$

This completes the proof. □

To obtain the L^2 error estimate for our general finite volume formulation (2.5), we assume that the bilinear form $a(v,w)$ satisfies the following equations.

Assumption 4. For any $v, w \in V(h)$,

$$a(v,w) = (\mathcal{A}\nabla_h v, \nabla_h w) + \sum_{T\in\mathcal{T}_h} \int_{\partial T} \mathcal{A}\nabla v \cdot \mathbf{n}(\gamma w - w)ds$$

$$+ \sum_{T\in\mathcal{T}_h} (\nabla \cdot \mathcal{A}\nabla v, w - \gamma w)_T. \tag{2.10}$$

For this reason, we shall take $\delta = -1$ in the following analysis.

THEOREM 2.2. *Let $u \in H^2(\Omega) \cap H_0^1(\Omega)$ and $u_h \in V_h$ be the solutions of* (1.1) *and* (2.5) *with $\delta = -1$, respectively. Assume that $\mathcal{A} \in W^{2,\infty}(\Omega)$ and that* (1.4), (1.5), (1.7), *and* (2.10) *hold. Then*

$$\|u - u_h\| \le Ch(|||u - u_h||| + h\|f\|_1).$$

Proof. Let $w \in H_0^1(\Omega) \cap H^2(\Omega)$ be the solution of the dual problem

$$-\nabla \cdot \mathcal{A}\nabla w = u - u_h \quad \text{in } \Omega, \tag{2.11}$$

$$w = 0 \quad \text{on } \partial\Omega, \tag{2.12}$$

so that the following estimate holds:

$$\|w\|_2 \le C\|u - u_h\|. \tag{2.13}$$

Let $w_I \in V_h$ be the usual *continuous* piecewise linear Lagrange interpolant of w, so that

$$|||w - w_I||| \le Ch|w|_2. \tag{2.14}$$

From (2.11) we deduce that

$$\begin{aligned}\|u - u_h\|^2 &= -(u - u_h, \nabla \cdot \mathcal{A}\nabla w)\\ &= (\mathcal{A}\nabla_h(u - u_h), \nabla_h w) - \sum_{T\in\mathcal{T}_h} \int_{\partial T} \mathcal{A}\nabla w \cdot \mathbf{n}(u - u_h)ds\\ &= (\mathcal{A}\nabla_h(u - u_h), \nabla_h w) - \sum_{e\in\mathcal{E}_h} (\{\mathcal{A}\nabla w\}, [\![u - u_h]\!])_e, \end{aligned} \tag{2.15}$$

where we have used (2.3) and the fact that $[\![\mathcal{A}\nabla w]\!]_e = 0$ on all interior edges e.

On the one hand, (2.10) implies

$$a(u - u_h, w_I) = (\mathcal{A}\nabla_h(u - u_h), \nabla_h w_I) + \sum_{T\in\mathcal{T}_h} (\mathcal{A}\nabla(u - u_h) \cdot \mathbf{n}, \gamma w_I - w_I)_{\partial T}$$

$$+ \sum_{T\in\mathcal{T}_h} (\nabla \cdot \mathcal{A}\nabla(u - u_h), w_I - \gamma w_I)_T, \tag{2.16}$$

and, on the other hand, it follows from (2.7) that

$$a(u - u_h, w_I) = \sum_{e\in\mathcal{E}_h} (\{\mathcal{A}\nabla w_I\}, [\![\gamma(u - u_h)]\!])_e. \tag{2.17}$$

Thus, subtracting (2.16) from the sum of (2.15) and (2.17), we have

$$\begin{aligned}\|u-u_h\|^2 &= (\mathcal{A}\nabla_h(u-u_h),\nabla_h(w-w_I)) - \sum_{T\in\mathcal{T}_h}(\nabla\cdot\mathcal{A}\nabla(u-u_h),w_I-\gamma w_I)_T \\ &\quad + \left(\sum_{e\in\mathcal{E}_h}(\{\mathcal{A}\nabla w_I\},[\![\gamma(u-u_h)]\!])_e - \sum_{e\in\mathcal{E}_h}(\{\mathcal{A}\nabla w\},[\![u-u_h]\!])_e\right) \\ &\quad - \sum_{T\in\mathcal{T}_h}(\mathcal{A}\nabla(u-u_h)\cdot\mathbf{n},\gamma w_I-w_I)_{\partial T} \\ &:= I_1+I_2+I_3+I_4. \end{aligned} \tag{2.18}$$

The four I terms can be estimated as follows. Using (2.14) and (2.13), we have

$$\begin{aligned} I_1 &= (\mathcal{A}\nabla_h(u-u_h),\nabla(w-w_I)) \le C|||u-u_h|||w-w_I|_1 \\ &\le Ch\|u-u_h\|\,|||u-u_h|||. \end{aligned}$$

As for the I_2 term, first it follows from (1.1), (1.4), (1.7), and (2.13) that

$$\begin{aligned}\sum_{T\in\mathcal{T}_h}(\nabla\cdot\mathcal{A}\nabla u,w_I-\gamma w_I)_T &= \sum_{T\in\mathcal{T}_h}(\bar f-f,w_I-\gamma w_I)_T \\ &\le Ch^2\|f\|_1\|u-u_h\|, \end{aligned}$$

where $\bar f$ is the average of f over each element. Next,

$$\begin{aligned}\sum_{T\in\mathcal{T}_h}(\nabla\cdot\mathcal{A}\nabla u_h,w_I-\gamma w_I)_T &= \sum_{T\in\mathcal{T}_h}(\nabla\cdot\mathcal{A}\nabla u_h-\overline{\nabla\cdot\mathcal{A}\nabla u_h},w_I-\gamma w_I)_T \\ &\le Ch||\mathcal{A}||_{2,\infty}|u_h|_{1,h}\|u-u_h\| \\ &\le Ch||\mathcal{A}||_{2,\infty}(|||u-u_h|||+||f||)\,\|u-u_h\|, \end{aligned} \tag{2.19}$$

where $\overline{\nabla\cdot\mathcal{A}\nabla u_h}$ is the average of $\nabla\cdot\mathcal{A}\nabla u_h$ over each element T.

For the I_3 term, using (1.5) and (2.13), we have

$$\begin{aligned}&\sum_{e\in\mathcal{E}_h}(\{\mathcal{A}\nabla w_I\},[\![\gamma(u-u_h)]\!])_e - \sum_{e\in\mathcal{E}_h}(\{\mathcal{A}\nabla w\},[\![(u-u_h)]\!])_e \\ &\quad= \sum_{e\in\mathcal{E}_h}(\{\mathcal{A}\nabla w_I\},[\![\gamma(u-u_h)]\!])_e - \sum_{e\in\mathcal{E}_h}(\{\mathcal{A}\nabla w\},[\![\gamma(u-u_h)]\!])_e \\ &\qquad+ \sum_{e\in\mathcal{E}_h}(\{\mathcal{A}\nabla w\},[\![\gamma(u-u_h)]\!])_e - \sum_{e\in\mathcal{E}_h}(\{\mathcal{A}\nabla w\},[\![(u-u_h)]\!])_e \\ &\quad= \sum_{e\in\mathcal{E}_h}(\{\mathcal{A}\nabla(w_I-w)\},[\![\gamma(u-u_h)]\!])_e \\ &\qquad- \sum_{e\in\mathcal{E}_h}(\{\mathcal{A}\nabla w-\overline{\mathcal{A}\nabla w}\},[\![(u-u_h)-\gamma(u-u_h)]\!])_e \\ &\quad:= J_1+J_2 \\ &\quad\le Ch|||u-u_h|||\,\|u-u_h\|, \end{aligned}$$

where $\overline{\mathcal{A}\nabla w}$ is the average of $\mathcal{A}\nabla w$ over each edge and the J terms are estimated as follows. In fact, the J_i terms can be estimated using the following easily derived trace inequality [12]: For $\phi\in H^1(T)$ and for an edge e of T with h_e the length of e,

$$\|\phi\|_e^2 \le C(h_e^{-1}|\phi|_{0,T}^2 + h_e|\phi|_{1,T}^2), \tag{2.20}$$

where C depends on the shape parameter of T such as the minimal angle of T in the triangular case. For instance,

$$\begin{aligned}
J_1 &= \sum_{e\in\mathcal{E}_h} (\{\mathcal{A}\nabla(w_I - w)\}, [\![\gamma(u-u_h)]\!])_e \\
&\le \sum_{e\in\mathcal{E}_h} |\{\mathcal{A}\nabla(w_I - w)\}|_{0,e} |[\![\gamma(u-u_h)]\!]|_{0,e} \\
&= \sum_{e\in\mathcal{E}_h} |\{\mathcal{A}\nabla(w_I - w)\}|_{0,e}\, h_e^{1/2} [\![\gamma(u-u_h)]\!]_e \\
&\le \sum_{e\in\mathcal{E}_h} h_e^{1/2} \left(h_e^{-1/2} |\{\mathcal{A}\nabla(w_I - w)\}|_T + h_e^{1/2} |\{\mathcal{A}\nabla(w_I - w)\}|_{1,T} \right) [\![\gamma(u-u_h)]\!]_e \\
&\le Ch ||\mathcal{A}||_{0,\infty} \|u-u_h\| \, |||u-u_h|||,
\end{aligned}$$

where we have used (2.20) in the last inequality. The term J_2 can be handled similarly.

For the I_4 term first observe that, for any matrix-valued function $\mathcal{M}$ such that $\mathcal{M}$ is constant on each $e \in \mathcal{E}_h$,

$$\begin{aligned}
\sum_{T\in\mathcal{T}_h} \int_{\partial T} \mathcal{M}\nabla(u-u_h)\cdot\mathbf{n}(\gamma w_I - w_I)ds &= \sum_{T\in\mathcal{T}_h} \int_{\partial T} \mathcal{M}\nabla u\cdot\mathbf{n}(\gamma w_I - w_I)ds \\
&\quad - \sum_{T\in\mathcal{T}_h} \int_{\partial T} \mathcal{M}\nabla u_h\cdot\mathbf{n}(\gamma w_I - w_I)ds \\
&= I_1 + I_2 = 0,
\end{aligned}$$

where $I_1 = 0$ due to $[\![\mathcal{M}\nabla u]\!] = 0$, and $[\![w_I - \gamma w_I]\!] = 0$ and $I_2 = 0$ due to the fact that $\mathcal{M}\nabla u_h \cdot \mathbf{n}$ is a constant on e and (1.5). Now define $\mathcal{M}$ so that on each $e \in \mathcal{E}_h$, $\mathcal{M} = \mathcal{A}(m)$, the value of $\mathcal{A}$ at the midpoint:

$$\begin{aligned}
|I_4| &= \left| \sum_{T\in\mathcal{T}_h} ((\mathcal{A}-\mathcal{M})\nabla(u-u_h)\cdot\mathbf{n}, \gamma w_I - w_I)_{\partial T} \right| \\
&\le Ch||A||_{1,\infty} \sum_{T\in\mathcal{T}_h} (|\nabla(u-u_h)\cdot\mathbf{n}|, |\gamma w_I - w_I|)_{\partial T} \\
&\le Ch||\mathcal{A}||_{1,\infty} |||u-u_h||| \, ||u-u_h||_0, \qquad (2.21)
\end{aligned}$$

where the last inequality was obtained via the trace inequality (2.20) as before.

Combining the above four estimates with (2.18), we obtain

$$\|u-u_h\| \le Ch(|||u-u_h||| + h\|f\|_1).$$

This completes the proof. □

The counterexamples in [15, 17] show that the assumption of $f \in H^1(\Omega)$ is necessary for finite volume methods.

3. Applications to finite volume and discontinuous finite volume methods. In this section, we will illustrate how our general theory can be applied to analyze different finite volume schemes.

3.1. Finite volume method with conforming trial functions. The finite volume discussed in this subsection is the classical finite volume method. For a given regular subdivision $\mathcal{T}_h$ of triangles, its dual partition $\mathcal{T}_h^*$ is the union of the convex hulls. These convex hulls in $\mathcal{T}_h^*$ are obtained by connecting the barycenters of the triangles and the midpoints of the edges of the triangles in $\mathcal{T}_h$ as shown in Figure 1.

The trial function space associated with $\mathcal{T}_h$ for the traditional finite volume method is defined as

$$V_h = \{v \in H_0^1(\Omega) : \ v|_T \in P_1(T) \ \forall T \in \mathcal{T}^h\},$$

with $V = H_0^1(\Omega)$ in (1.2). The test function space is defined as in (1.3).

Let $\mathcal{N}$ be a set containing all of the interior nodal points associated with the partition $\mathcal{T}^h$. The operator $\gamma : V(h) \to Q_h$ is defined by

$$\gamma v(x) \equiv \sum_{P \in \mathcal{N}} v(P)\chi_P(x) \quad \forall x \in \Omega, \tag{3.1}$$

where χ_P is the characteristic function of the dual element K_P^* associated with the node P. It can be easily verified that γ defined in (3.1) satisfies (1.4)–(1.7).

The traditional conforming finite volume method is to find $u_h \in V_h$ such that for any $v \in V_h$

$$a(u_h, v) = (f, \gamma v). \tag{3.2}$$

The bilinear form $A(v, w)$ in (2.5) reduces to $a(v, w)$ and

$$a(u, v) = -\sum_{K \in \mathcal{T}_h^*} \int_{\partial K} \mathcal{A}\nabla u \cdot \mathbf{n}\gamma v ds.$$

LEMMA 3.1. *For any* $v, w \in V(h)$,

$$\begin{aligned} a(v, w) = (\mathcal{A}\nabla v, \nabla w) &+ \sum_{T \in \mathcal{T}_h} \int_{\partial T} \mathcal{A}\nabla v \cdot \mathbf{n}(\gamma w - w) ds \\ &+ \sum_{T \in \mathcal{T}_h} (\nabla \cdot \mathcal{A}\nabla v, w - \gamma w)_T. \end{aligned} \tag{3.3}$$

Proof. Equation (3.3) appeared in [12, 15, 24], and for completeness we include a short proof here. For ease of proof, a typical primal triangle in Figure 1 is isolated and indexed as in Figure 3. For $j = 1, 2, 3$, let $\square_j$ denote the quadrilaterals formed by the four corner nodes Q, M_j, P_{j+1}, M_{j+1} as shown in Figure 3; when out of bound we use $M_4 = M_1$ and $P_4 = P_1$. Using the divergence theorem on each quadrilateral,

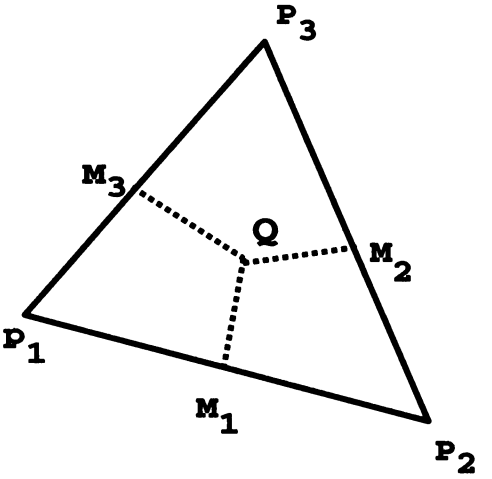

FIG. 3. *Partial primal and dual grids for integration.*

we have

$$
\begin{aligned}
a(v,w) &= -\sum_{T\in\mathcal{T}_h}\sum_{j=1}^{3}\int_{M_{j+1}QM_j}\mathcal{A}\nabla v\cdot\mathbf{n}\gamma w ds\\
&= \sum_{T\in\mathcal{T}_h}\sum_{j=1}^{3}\int_{M_jP_{j+1}M_{j+1}}\mathcal{A}\nabla v\cdot\mathbf{n}\gamma w ds - \sum_{T\in\mathcal{T}_h}\sum_{\square_j}(\nabla\cdot\mathcal{A}\nabla v,\gamma w)\\
&= \sum_{T\in\mathcal{T}_h}\sum_{j=1}^{3}\int_{M_jP_{j+1}M_{j+1}}\mathcal{A}\nabla v\cdot\mathbf{n}(\gamma w - w)ds + \sum_{T\in\mathcal{T}_h}\int_{\partial T} w\mathcal{A}\nabla v\cdot\mathbf{n}ds\\
&\quad - \sum_{T\in\mathcal{T}_h}\sum_{\square_j}(\nabla\cdot\mathcal{A}\nabla v,\gamma w)\\
&= \sum_{T\in\mathcal{T}_h}\int_{\partial T}\mathcal{A}\nabla v\cdot\mathbf{n}(\gamma w - w)ds + \sum_{T\in\mathcal{T}_h}(\mathcal{A}\nabla v,\nabla w)_T + \sum_{T\in\mathcal{T}_h}(\nabla\cdot\mathcal{A}\nabla v, w)_T\\
&\quad - \sum_{T\in\mathcal{T}_h}\sum_{\square_j}(\nabla\cdot\mathcal{A}\nabla v,\gamma w)\\
&= (\mathcal{A}\nabla v,\nabla w) + \sum_{T\in\mathcal{T}_h}\int_{\partial T}\mathcal{A}\nabla v\cdot\mathbf{n}(\gamma w - w)ds + \sum_{T\in\mathcal{T}_h}(\nabla\cdot\mathcal{A}\nabla v, w-\gamma w)_T. \qquad \square
\end{aligned}
$$

This lemma implies that Assumption 3 holds: The boundedness of $a(v,w)$ is straightforward. For the proof of coercivity (2.9) on V_h, notice the following. First of all, $|\!|\!|v|\!|\!| = |v|_{1,h}$, and so $C|\!|\!|v|\!|\!| \le (\mathcal{A}\nabla v,\nabla v)$ for all $v\in V_h$. The last two terms in the right side of (3.3) are the $O(h|v|_{1,h}^2)$ term when $v=w$. In fact, just as in estimating the I_4 term of (2.21), we have

$$
\sum_{T\in\mathcal{T}_h}\int_{\partial T}\mathcal{A}\nabla v\cdot\mathbf{n}(\gamma v - v)ds \le Ch\|A\|_{1,\infty}|v|_{1,h}^2
$$

and

$$\sum_{T\in\mathcal{T}_h}(\nabla\cdot\mathcal{A}\nabla v, v-\gamma v)_T = \sum_{T\in\mathcal{T}_h}(\nabla\cdot\mathcal{A}\cdot\nabla v, v-\gamma v)_T \le Ch\|A\|_{1,\infty}|v|^2_{1,h}, \tag{3.4}$$

where $\nabla\cdot\mathcal{A}$ is the vector obtained by applying the divergence rowwise. Thus for h small enough we have the coercivity. Note that this last term could be handled like (2.19), but this would require $\mathcal{A}$ to be in $W^{2,\infty}$, which is unnecessary.

Applying Theorems 2.1 and 2.2, we have the following results.

THEOREM 3.1. *If $u \in H_0^1(\Omega)\cap H^2(\Omega)$ and $f \in H^1(\Omega)$, then*

$$|||u-u_h||| \le Ch\|u\|_2,$$
$$\|u-u_h\| \le Ch^2(\|u\|_2+\|f\|_1),$$

where the L^2 estimate requires $\mathcal{A}\in W^{2,\infty}(\Omega)$.

The same conclusions hold for the conforming bilinear trial function case [9], and we omit the details.

3.2. Finite volume method with nonconforming trial functions. For a given regular triangulation $\mathcal{T}_h$, its dual partition $\mathcal{T}_h^*$ is the union of quadrilaterals. Each quadrilateral in $\mathcal{T}_h^*$ is made up of two subtriangles which share a common edge (see Figure 1). These subtriangles are formed by connecting the barycenter and the three corners of the triangles.

The trial function space associated with $\mathcal{T}_h$ for the nonconforming finite volume method is defined as

$$\begin{aligned} V_h = \{&v\in L^2(\Omega):\ v|_T\in P_1(T)\ \ \forall T\in\mathcal{T}_h,\\ &\text{is continuous at the midpoint of } e\in\mathcal{E}_h^0\\ &\text{and is zero at the midpoint of boundary edges } e \text{ on } \partial\Omega\}. \end{aligned}$$

The test function space is defined as in (1.3).

Let $\mathcal{M}$ be a set containing all of the midpoints of the interior edges associated with the triangulation $\mathcal{T}^h$. The operator $\gamma: V(h)\to Q_h$ is defined by

$$\gamma v(x) \equiv \sum_{P\in\mathcal{M}} v(P)\chi_P(x) \quad \forall x\in\Omega, \tag{3.5}$$

where χ_P is the characteristic function of dual element K_P^* associated with the node P. The mapping γ satisfies Assumptions 1 and 2 (see [8]). Finite volume methods using the above nonconforming trial functions were considered in [8, 6].

Our version [8] is to find $u_h\in V_h$ such that for any $v\in V_h$

$$a(u_h,v) = (f,\gamma v). \tag{3.6}$$

The bilinear form $A(v,w)$ in (2.5) reduces to $a(v,w)$ and

$$a(u,v) = -\sum_{K\in\mathcal{T}_h^*}\int_{\partial K}\mathcal{A}\nabla u\cdot\mathbf{n}\gamma v ds.$$

LEMMA 3.2. *For any $v,w\in V(h)$,*

$$\begin{aligned} a(v,w) = (\mathcal{A}\nabla_h v,\nabla_h w) &+ \sum_{T\in\mathcal{T}_h}\int_{\partial T}\mathcal{A}\nabla v\cdot\mathbf{n}(\gamma w - w)ds\\ &+ \sum_{T\in\mathcal{T}_h}(\nabla\cdot\mathcal{A}\nabla v, w-\gamma w)_T. \end{aligned}$$

Proof. See Lemma 3.2 in [24]. □

Using the above lemma, as before we can prove that (2.8) and (2.9) hold easily. Then we have the following estimates.

THEOREM 3.2. *If* $u \in H_0^1(\Omega) \cap H^2(\Omega)$ *and* $f \in H^1(\Omega)$, *then*

$$|||u - u_h||| \le Ch\|u\|_2$$
$$\|u - u_h\| \le Ch^2(\|u\|_2 + \|f\|_1),$$

where the L^2 *estimate requires* $\mathcal{A} \in W^{2,\infty}$.

The same conclusions hold for the finite volume method [10] using the rotated bilinear trial functions, i.e., the nonconforming Q_1 elements on rectangular grids [23]. We omit the details here.

3.3. Finite volume method with totally discontinuous trial functions. The finite volume method using totally discontinuous trial functions was first proposed in [24].

Let $\mathcal{T}_h$ be a quasiuniform triangulation of Ω. We define the dual partition $\mathcal{T}_h^*$ of $\mathcal{T}_h$ for the test function space as follows. We divide each $T \in \mathcal{T}_h$ into three triangles by connecting the barycenter and the three corners of the triangle as shown in Figure 2. Let $\mathcal{T}_h^*$ consist of all of these triangles T_j, $j = 1, 2, 3$.

We define the finite dimensional space associated with $\mathcal{T}_h$ for the trial functions as

$$V_h = \{v \in L^2(\Omega) : \ v|_T \in P_1(T) \ \forall T \in \mathcal{T}_h\}. \tag{3.7}$$

The test function space is defined as in (1.3). The operator $\gamma : V(h) \to Q_h$ is defined as

$$\gamma v|_T = \frac{1}{h_e}\int_e v|_T ds \quad \forall T \in \mathcal{T}_h, \tag{3.8}$$

where h_e is the length of the edge e. The operator γ satisfies (1.4)–(1.7) (see [25]).

The discontinuous finite volume method is to find $u_h \in V_h$ such that

$$A(u_h, v) = (f, \gamma v) \qquad \forall v \in V_h. \tag{3.9}$$

LEMMA 3.3. *For any* $v, w \in V(h)$,

$$a(v, w) = (\mathcal{A}\nabla_h v, \nabla_h w) + \sum_{T\in\mathcal{T}_h}\int_{\partial T} \mathcal{A}\nabla v \cdot \mathbf{n}(\gamma w - w)ds$$
$$+ \sum_{T\in\mathcal{T}_h} (\nabla \cdot \mathcal{A}\nabla v, w - \gamma w)_T. \tag{3.10}$$

Proof. See Lemma 2.1 in [25]. □

Using the above lemma, one can prove coercivity and boundedness.

LEMMA 3.4. *There is a constant* C *independent of* h *such that*

$$A(v, v) \ge C|||v|||^2 \qquad \forall v \in V_h \tag{3.11}$$

for any positive α *if* $\delta = 1$ *and for* α *larger enough if* $\delta = -1$.

Proof. See Lemma 2.2 in [25]. □

LEMMA 3.5. *For $v, w \in V(h)$, we have*

$$A(v,w) \leq C|||v|||\,|||w|||. \tag{3.12}$$

Proof. See Lemma 2.3 in [25]. □

Since all of the conditions for Theorems 2.1 and 2.2 are satisfied, we have the following error estimates for the discontinuous finite volume method.

THEOREM 3.3. *If $u \in H_0^1(\Omega) \cap H^2(\Omega)$ and $f \in H^1(\Omega)$, then*

$$|||u - u_h||| \leq Ch\|u\|_2,$$
$$\|u - u_h\| \leq Ch^2(\|u\|_2 + \|f\|_1).$$

We point out that the above L^2 estimate was not obtained in [25].

REFERENCES

[1] D. N. ARNOLD, F. BREZZI, B. COCKBURN, AND L. D. MARINI, *Unified analysis of discontinuous Galerkin methods for elliptic problems*, SIAM J. Numer. Anal., 39 (2002), pp. 1749–1779.

[2] R. E. BANK AND D. J. ROSE, *Some error estimates for the box method*, SIAM J. Numer. Anal., 24 (1987), pp. 777–787.

[3] S. BRENNER AND R. SCOTT, *Mathematical Theory of Finite Element Methods*, Springer, New York, 2002.

[4] F. BREZZI, B. COCKBURN, L. D. MARINI, AND E. SULI, *Stabilization mechanisms in discontinuous Galerkin finite element methods*, Comput. Methods Appl. Mech. Engrg., 195 (2006), pp. 3293–3310.

[5] Z. CAI, J. MANDEL, AND S. MCCORMICK, *The finite volume element method for diffusion equations on general triangulations*, SIAM J. Numer. Anal., 28 (1991), pp. 392–402.

[6] P. CHATZIPANTELIDIS, *Finite volume methods for elliptic PDE's: A new approach*, Math. Model. Numer. Anal., 36 (2002), pp. 307–324.

[7] Z. CHEN, G. HUAN, AND Y. MA, *Computational Methods for Multiphase Flows in Porous Media*, SIAM, Philadelphia, PA, 2006.

[8] S. H. CHOU, *Analysis and convergence of a covolume method for the generalized Stokes problem*, Math. Comp., 66 (1997), pp. 85–104.

[9] S. H. CHOU AND D. Y. KWAK, *Analysis and convergence of a MAC scheme for the generalized Stokes problem*, Numer. Methods Partial Differential Equations, 13 (1997), pp. 147–162.

[10] S. H. CHOU AND D. Y. KWAK, *A covolume method based on rotated bilinears for the generalized Stokes problem*, SIAM J. Numer. Anal., 35 (1998), pp. 494–507.

[11] S. H. CHOU, D. Y. KWAK, AND K. Y. KIM, *Mixed finite volume methods on non-staggered quadrilateral grids for elliptic problems*, Math. Comp., 72 (2003), pp. 525–539.

[12] S. H. CHOU AND Q. LI, *Error estimates in L^2, H^1, and L^∞ in covolume methods for elliptic and parabolic problems, a unified approach*, Math. Comp., 69 (2000), pp. 103–120.

[13] S. H. CHOU AND P. S. VASSILEVSKI, *A general mixed covolume framework for constructing conservative schemes for elliptic problems*, Math. Comp., 68 (1999), pp. 991–1011.

[14] M. CROUZEIX AND P. A. RAVIART, *Conforming and nonconforming finite element methods for solving the stationary Stokes equation* I, RAIRO Anal. Numer., 7 (1973), pp. 33–76.

[15] R. E. EWING, T. LIN, AND Y. LIN, *On the accuracy of the finite volume element method based on piecewise linear polynomials*, SIAM J. Numer. Anal., 39 (2002), pp. 1865–1888.

[16] R. EYMARD, GALLOUET, AND R. HERBIN, *Finite volume methods*, Handbook of Numerical Analysis VII, P. G. Ciarlet and J. L. Lions, eds., North-Holland, 2000, pp. 713–1020.

[17] J. HUANG AND S. XI, *On the finite volume elment method for general self-adjoint elliptic problems*, SIAM J. Numer. Anal., 35 (1998), pp. 1762–1774.

[18] R. D. LAZAROV, I. D. MISHEV, AND P. S. VASSILEVSKI, *Finite volume methods for convection-diffusion problems*, SIAM J. Numer. Anal., 33 (1996), pp. 31–55.

[19] C. LIU AND S. F. MCCORMICK, *The finite volume-element method for planar cavity flow*, in Proceedings of the 11th International Conference on Numerical Methods in Fluid Dynamics, Williamsburg, 1988, Springer, Berlin, 1988.

[20] S. MCCORMICK, *Multilevel Adaptive Methods for Partial Differential Equations*, SIAM, Philadelphia, 1989.

[21] R. A. NICHOLAIDES, T. A. PORSCHING, AND C. A. HALL, *Covolume methods in computational fluid dynamics*, Comput. Dyn. Rev., M. Hafez and K. Oshima, eds., 1995, Wiley, New York, pp. 279–299.
[22] Z. Y. CHEN, R. H. LI, AND W. WU, *Generalized Difference Methods for Differential Equations*, Marcel Dekker, New York, 2000.
[23] R. RANNACHER AND S. TUREK, *Simple nonconforming quadrilateral Stokes element*, Numer. Methods Partial Differential Equations, 8 (1992), pp. 97–111.
[24] X. YE, *On the relationship between finite volume and finite element methods applied to the Stokes equations*, Numer. Methods Partial Differential Equations, 17 (2001), pp. 440–453.
[25] X. YE, *A new discontinuous finite volume method for elliptic problems*, SIAM J. Numer. Anal., 42 (2004), pp. 1062–1072.