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MIXED COVOLUME METHODS ON RECTANGULAR GRIDS FOR ELLIPTIC PROBLEMS*

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Abstract. We consider a covolume method for a system of first order PDEs resulting from the mixed formulation of the variable-coefficient-matrix Poisson equation with the Neumann boundary condition. The system may be used to represent the Darcy law and the mass conservation law in anisotropic porous media flow. The velocity and pressure are approximated by the lowest order Raviart–Thomas space on rectangles. The method was introduced by Russell [*Rigorous Block-centered Discretizations on Irregular Grids: Improved Simulation of Complex Reservoir Systems*, Reservoir Simulation Research Corporation, Denver, CO, 1995] as a control-volume mixed method and has been extensively tested by Jones [*A Mixed Finite Volume Elementary Method for Accurate Computation of Fluid Velocities in Porous Media*, University of Colorado at Denver, 1995] and Cai et al. [*Computational Geosciences*, 1 (1997), pp. 289–345]. We reformulate it as a covolume method and prove its first order optimal rate of convergence for the approximate velocities as well as for the approximate pressures.

Key words. covolume method, mixed method, finite volume element, error estimate, porous media

AMS subject classifications. 65N30, 65N22, 65F10

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1. Introduction.

Consider the Poisson equation in an axiparallel domain $\Omega \subset \mathbb{R}^2$

$$\begin{cases} -\nabla \cdot \mathcal{K}\nabla p = f & \text{in } \Omega, \\ \mathcal{K}\nabla p \cdot \mathbf{n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\mathcal{K} = \mathcal{K}(\mathbf{x}) = \mathrm{diag}(\tau_1^{-1}(\mathbf{x}), \tau_2^{-1}(\mathbf{x}))$ is a positive definite diagonal matrix function and its entries are bounded from below and above by positive constants. Furthermore, we shall assume that τ_1, τ_2 are locally Lipschitz.

Let us introduce a new variable $\mathbf{u} = -\mathcal{K}\nabla p$ and write the above equation as the system of first order partial differential equations

$$\begin{cases} \mathcal{K}^{-1}\mathbf{u} = -\nabla p & \text{in } \Omega, \\ \mathrm{div}\mathbf{u} = f & \text{in } \Omega, \end{cases} \tag{1.2}$$

with the boundary condition $\mathbf{u} \cdot \mathbf{n} = 0$ on $\partial\Omega$. This system can be interpreted as modeling an incompressible single phase flow in a reservoir, ignoring gravitational effects. The matrix $\mathcal{K}$ is the mobility κ/μ, the ratio of permeability tensor to viscosity of the fluid, $\mathbf{u}$ is the Darcy velocity, and p is the pressure. The first equation is the Darcy law and the second represents conservation of mass with f standing for a source or sink term. Since κ is in general discontinuous due to different rock formations, separating the Darcy law from the second order equation and discretizing it directly

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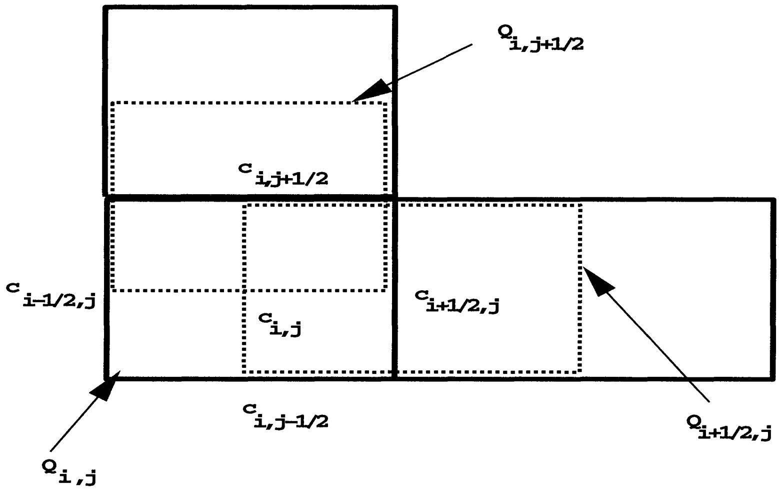

FIG. 1.1. *Primal and dual domains*

together with the mass conservation may lead to a better numerical treatment on the velocity than just computing it from the pressure via the Darcy law [1, 12].

The associated weak formulation of our first order system is the following: Find $(\mathbf{u}, p) \in \mathbf{H}_0 \times L_0^2$ such that

$$
\begin{array}{rclcl}
(\mathcal{K}^{-1}\mathbf{u}, \mathbf{v}) & = & (p, \operatorname{div} \mathbf{v}) & \forall & \mathbf{v} \in \mathbf{H}_0, \\
(\operatorname{div}\mathbf{u}, q) & = & (f, q) & \forall & q \in L_0^2,
\end{array}
\tag{1.3}
$$

where $\mathbf{H}_0 := \mathbf{H}(\operatorname{div}; \Omega) \cap \{\mathbf{u} \cdot \mathbf{n} = 0\}$ and $L_0^2 := \{q \in L^2(\Omega) : \int_\Omega q dx = 0\}$. The space $\mathbf{H}(\operatorname{div}; \Omega)$ is the set of all vector-valued functions $\mathbf{w} \in L^2(\Omega)^2$ such that $\operatorname{div} \mathbf{w} \in L^2(\Omega)$.

We will adapt a covolume methodology for the generalized Stokes problem [3] to approximate this system. The basic idea of creating a covolume method is to find a good combination of the finite volume method and the MAC (marker and cell) [11] placements of flow variables. (A balanced survey of the covolume method literature up to 1995 is in Nicolaides, Porsching, and Hall [13].) In the MAC scheme, the pressure variable is assigned to the centers of the rectangular volumes, and the normal components of the velocity or fluxes are assigned to the edges of the rectangular volumes.

More specifically, let $\mathcal{Q}_h = \{Q_{i,j}\}$ be a partition of the domain Ω into a union of rectangles $Q_{i,j}$ with centers $c_{i,j}$ (see Figure 1.1). The subindices $\{i+1, j\}, \{i-1, j\}$, $\{i, j+1\}$, and $\{i, j-1\}$ are assigned to the eastern, western, northern, and southern adjacent rectangles, respectively, if they exist. Given $Q_{i,j}$, the two midpoints of its vertical edges are denoted as $c_{i\pm1/2,j}$ and the two midpoints of its horizontal edges as $c_{i,j\pm1/2}$. Let $c_{i,j} = (x_i, y_j)$ and $c_{i+1/2,j} = (x_{i+1/2}, y_j)$ etc., define

$$
\begin{aligned}
Q_{i+1/2,j} &:= \big([x_i, x_{i+1}] \times [y_{j-1/2}, y_{j+1/2}]\big) \cap \overline{\Omega}, \\
Q_{i,j+1/2} &:= \big([x_{i-1/2}, x_{i+1/2}] \times [y_j, y_{j+1}]\big) \cap \overline{\Omega},
\end{aligned}
$$

and

$$
Q_{i,j} := [x_{i-1/2}, x_{i+1/2}] \times [y_{j-1/2}, y_{j+1/2}].
$$

Since the approximate pressure is assigned at the center of $Q_{i,j}$, it is natural to assume that as a function on Ω it is piecewise constant with respect to the primal partition $\{Q_{i,j}\}$. The unknown approximate pressure p_h at the center of Q_{ij} can then be determined by integrating the mass conservation equation over $Q_{i,j}$.

The normal approximate velocity is assumed to be constant along any edge. There are several ways [1, 4, 6, 10] to exactly or nearly accomplish this; here we will use the one proposed in [1, 12], i.e., use the lowest order Raviart–Thomas spaces for the approximate velocity field. Within each $Q_{i,j}$ the horizontal component of the velocity is linear in x and constant in y, whereas the vertical component is linear in y and constant in x. Thus we have four degrees of freedom assigned at midpoints of edges. For example, with the eastern vertical edge of $Q_{i,j}$, we associate one unknown, the horizontal velocity component (normal flux); the accompanying equation is taken by integrating the first component of the vector equation $(1.2)_1$ over $Q_{i+1/2,j}$. Similarly, to determine the unknown at a nonborder northern edge, we integrate $(1.2)_2$ over $Q_{i,j+1/2}$. In other words, if we write the velocity field $\mathbf{u}_h = (u_h, v_h)$, then $Q_{i,j+1/2}$ is for the determination of v_h and $Q_{i+1/2,j}$ is for u_h. We will sometimes call $Q_{i+1/2,j}$ ($Q_{i,j+1/2}$) a u-volume (v-volume). These volumes are also called the covolumes of $Q_{i,j}$ in the literature [13].

Throughout this paper the primal partition $\mathcal{Q}_h = \{Q_{ij}\}$ is *quasi-regular*, i.e., there exist positive constants C_1 and C_2 independent of h such that

$$C_1 h^2 \le \text{ area}\{Q_{i,j}\} \le C_2 h^2 \quad \forall \quad Q_{ij} \in \mathcal{Q}_h, \tag{1.4}$$

where $h := \max_{i,j}\{h^x_{i,j}, h^y_{i,j}\}$, $h^x_{i,j}, h^y_{i,j}$ are, respectively, the width and height of $Q_{i,j}$. Now define the following two spaces:

$$\begin{aligned} \mathbf{Y}_h := \{(u_h, v_h) : u_h \in L^2(\Omega) &\text{ is piecewise constant on } u\text{-volumes},\\ v_h \in L^2(\Omega) &\text{ is piecewise constant on } v\text{-volumes}\}\\ \bigcap \{(u_h, v_h) : u_h = 0 &\text{ on border } u\text{-volumes}, v_h = 0 \text{ on border } v\text{-volumes}\}, \end{aligned}$$

and

$$\begin{aligned} \mathbf{H}_h := \{(u_h, v_h) \in \mathbf{H}_0 : u_h(x,y) &= a + bx,\\ v_h(x,y) &= c + dy \text{ on } Q_{i,j} \in \mathcal{Q}_h\}. \end{aligned}$$

The *trial* space $\mathbf{H}_h$ is the lowest order Raviart–Thomas space, and $\mathbf{Y}_h$ is the *test* space used to pick out the control volumes in engineering applications. For the pressure space, define

$$L_h := \{q_h \in L^2_0 : q_h \text{ is constant over } Q_{ij} \in \mathcal{Q}_h\}.$$

We now describe the above processes more abstractly, since our purpose is to prove convergence. For computational results and more applications see [1, 12]. The standard mixed finite element method for (1.1) deals with the primal grid only: Find $(\tilde{\mathbf{u}}_h, \tilde{p}_h) \in \mathbf{H}_h \times L_h$ such that

$$\begin{array}{rcll} (\mathcal{K}^{-1}\tilde{\mathbf{u}}_h, \mathbf{v}_h) - (\operatorname{div} \mathbf{v}_h, \tilde{p}_h) &=& 0 & \forall \quad \mathbf{v}_h \in \mathbf{H}_h,\\ (\operatorname{div} \tilde{\mathbf{u}}_h, q_h) &=& (f, q_h) & \forall \quad q_h \in L_h. \end{array} \tag{1.5}$$

In contrast, the present method deals with three grids. First, we define an analogue of $-(\mathrm{div}\mathbf{v}_h, \tilde{p}_h)$ in (1.5). Note that for smooth $\mathbf{v} = (v^1, v^2)$ and p

$$
\begin{aligned}
-(\mathrm{div}\mathbf{v}, p) &= -(\partial v^1/\partial x, p) - (\partial v^2/\partial y, p) \\
&= -\sum_{i,j} \int_{Q_{i+1/2,j}} \frac{\partial v^1}{\partial x}\, p\, dxdy - \sum_{i,j} \int_{Q_{i,j+1/2}} \frac{\partial v^2}{\partial y}\, p\, dxdy \\
&= -\sum_{i,j} \int_{\partial Q_{i+1/2,j}} (v^1, 0)^t \cdot \mathbf{n} p\, ds - \sum_{i,j} \int_{\partial Q_{i,j+1/2}} (0, v^2)^t \cdot \mathbf{n} p ds. \qquad (1.6)
\end{aligned}
$$

With this in mind we define the bilinear form $b : \mathbf{Y}_h \times L_h \to \mathbb{R}$:

$$
\begin{aligned}
b(\mathbf{v}_h, p_h) \ := \ & -\sum_{i,j} \int_{\partial Q_{i+1/2,j}} (v_h^1(c_{i+1/2,j}), 0)^t \cdot \mathbf{n} p_h ds \\
& -\sum_{i,j} \int_{\partial Q_{i,j+1/2}} (0, v_h^2(c_{i,j+1/2}))^t \cdot \mathbf{n} p_h ds
\end{aligned} \qquad (1.7)
$$

and the bilinear forms $a : \mathbf{H}_h \times \mathbf{Y}_h \to \mathbb{R}$, $c : \mathbf{H}_h \times L_h \to \mathbb{R}$:

$$
a(\mathbf{u}_h, \mathbf{v}_h) \ := \ \int_\Omega \mathcal{K}^{-1} \mathbf{u}_h \cdot \mathbf{v}_h dx\, dy, \qquad (1.8)
$$

$$
\begin{aligned}
c(\mathbf{u}_h, q_h) \ &:= \ -\sum q_h(c_{i,j}) \int_{Q_{i,j}} \mathrm{div}\mathbf{u}_h dx\, dy \qquad (1.9) \\
&= \ -\int_\Omega q_h \mathrm{div}\mathbf{u}_h dx\, dy.
\end{aligned}
$$

Note that the form $a(\cdot,\cdot)$ can be extended to $L^2(\Omega) \times L^2(\Omega)$ and will be also used as such later. In addition, define the *transfer* operator $\gamma_h : \mathbf{H}_h \to \mathbf{Y}_h$:

$$
\begin{aligned}
\gamma_h \mathbf{w}_h &:= (\gamma_h u_h, \gamma_h v_h), \qquad \mathbf{w}_h = (u_h, v_h) \\
&:= \left(\sum_{i,j} u_h(c_{i+1/2,j}) \chi_{i+1/2,j}, \sum_{i,j} v_h(c_{i,j+1/2}) \chi_{i,j+1/2} \right), \qquad (1.10)
\end{aligned}
$$

where $\chi_{i+1/2,j}$ and $\chi_{i,j+1/2}$ are the characteristic functions of $Q_{i+1/2,j}$ and $Q_{i,j+1/2}$, respectively. Note that we used the same notation γ_h in the componentwise definition and that γ_h is onto. Now let $\chi_j(y)$ and $\bar{\chi}_i(x)$ be the characteristic functions of the intervals $[y_{j-1/2}, y_{j+1/2}]$ and $[x_{i-1/2}, x_{i+1/2}]$, respectively. Recall that the space $\mathbf{H}_h$ is spanned by the functions $\{(\phi_{i+1/2,j}(x)\chi_j(y), 0)\}$ and $\{(0, \psi_{i,j+1/2}(y)\bar{\chi}_i(x)\}$ where $\phi_{i+1/2,j}$ and $\psi_{i,j+1/2}$ are the usual hat functions associated with midpoints of the interior vertical and horizontal edges, respectively. In Figure 1.2, we show a typical action of the transfer operator on the basis functions. The top figure is the basis function $\mathbf{w}_h = (\phi_{i+1/2,j}(x)\chi_j(y), 0)$ based at the common edge with center point $c_{i+1/2,j}$. The bottom figure is the image of $\mathbf{w}_h$ under γ_h, whose x-component is the characteristic function of the dotted covolume and whose y-component is zero. The action on $(0, \psi_{i,j+1/2}(y)\bar{\chi}_i(x))$ can be shown similarly.

REMARK 1.1. *A remark is in order here concerning the bilinear form b. Although we motivated its definition using* (1.6) *with smooth functions, notice that b has in its first argument a nonsmooth test function in $\mathbf{Y}_h$. We shall show in Lemma* 2.1 *that $b(\gamma_h \mathbf{w}_h, p_h) = -c(\mathbf{w}_h, p_h)$. In other words, it is $b(\gamma_h\cdot,\cdot)$, not $b(\cdot,\cdot)$, that is related to the divergence term in* (1.5).

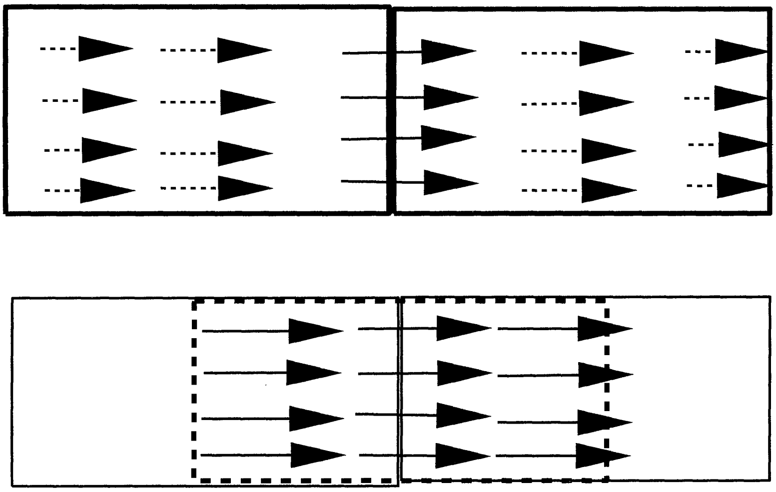

FIG. 1.2. *The action of the transfer operator* $\gamma_h : \mathbf{H}_h \to \mathbf{Y}_h$ *on a basis function. The top figure is the basis vector function with the support in the two primal volumes. It is constant in the y–direction and has value one along the common edge, zero along the boundary edges, and linear in between. The bottom figure is its image in* $\mathbf{Y}_h$*, which has constant value* $(1,0)$ *throughout the dotted covolume and zero elsewhere.*

Thus, the covolume method we consider is this: Find $\{\mathbf{u}_h, p_h\} \in \mathbf{H}_h \times L_h$ such that

$$\begin{array}{rcll} a(\mathbf{u}_h, \gamma_h \mathbf{w}_h) - b(\gamma_h \mathbf{w}_h, p_h) & = & 0 & \forall \quad \mathbf{w}_h \in \mathbf{H}_h, \\ -c(\mathbf{u}_h, q_h) & = & (f, q_h) & \forall \quad q_h \in L_h. \end{array} \tag{1.11}$$

Here the substitution of $\gamma_h \mathbf{w}_h$ for a test function $\mathbf{v}_h \in \mathbf{Y}_h$ is due to the surjectiveness of the operator γ_h. This simple observation turns the original Petrov–Galerkin statement into a standard Galerkin one. It can be easily checked that this formulation reduces to that of Cai et al. [1] once the characteristic functions of u- and v-volumes are chosen as basis functions in representing it as a linear system. We can reformulate (1.11) into a standard saddle point problem by further introducing

$$A(\mathbf{u}, \mathbf{v}) := a(\mathbf{u}, \gamma_h \mathbf{v}) = (\mathcal{K}^{-1}\mathbf{u}, \gamma_h \mathbf{v}) \quad \mathbf{u}, \mathbf{v} \in \mathbf{H}_h$$

and

$$B(\mathbf{w}_h, q_h) := b(\gamma_h \mathbf{w}_h, q_h)$$

so that problem (1.11) becomes

$$\begin{array}{rcll} A(\mathbf{u}_h, \mathbf{w}_h) - B(\mathbf{w}_h, p_h) & = & 0 & \forall \quad \mathbf{w}_h \in \mathbf{H}_h, \\ -c(\mathbf{u}_h, q_h) & = & (f, q_h) & \forall \quad q_h \in L_h. \end{array} \tag{1.12}$$

In Lemma 2.1, we show that $B = -c$ and hence

$$A(\mathbf{u}_h, \mathbf{w}_h) - B(\mathbf{w}_h, p_h) = 0 \qquad \forall \quad \mathbf{w}_h \in \mathbf{H}_h, \tag{1.13}$$

$$B(\mathbf{u}_h, q_h) = (f, q_h) \qquad \forall \quad q_h \in L_h. \tag{1.14}$$

Note that the above system is in standard form. Nevertheless, the standard mixed method analysis cannot be used here. This is so because the original PDE cannot be put into the same form—the transfer operator γ_h in the definition of the bilinear form A cannot be extended to the space $\mathbf{H}(\text{div};\Omega)$. However, on closer examination we see that the standard mixed method (1.5) for the Poisson equation (1.2) differs from the mixed covolume method (1.13)–(1.14) only in the bilinear form A. Thus we can treat the covolume method as one resulting from a "variational crime" of the standard mixed method. A careful analysis of the transfer operator γ_h in connection to this deviation then leads to our error estimate in Theorem 3.1 which demonstrates the first order error estimates in both the velocity and the pressure. The starting point of the proof is a good error equation (cf. (3.9) below) that plays the role of Cea's lemma in the standard finite element analysis. This methodology was initiated in Chou [3] for the generalized Stokes problem on triangular grids, in Chou and Kwak [6] for the same problem on rectangular grids, and in Chou and Li [5] for the "point-centered" or vertex-centered schemes for the variable-coefficient Poisson equation. A general framework for constructing and analyzing mixed covolume methods for convection-diffusion equations is Chou and Vassilevski [7]. The present paper also introduces some new techniques to overcome difficulties in dealing with the space $\mathbf{H}(\text{div};\Omega)$ and the variable-coefficient (mobility) matrix in the covolume setting. Other methodology of proving convergence for the finite volume element method can be found in Cai and McCormick [2].

2. Saddle point formulation. In this section we prove some preliminary lemmas. Let $||\cdot||_j, j=0,1$ denote the usual L^2 and H^1 norms, respectively, and let

$$||\mathbf{u}||^2_{H(\text{div})} := ||\mathbf{u}||_0^2 + ||\text{div}\mathbf{u}||_0^2.$$

We also use $||\cdot||$ for the L^2 norm when there is no confusion. The symbol C will be used as a generic positive constant independent of h and may have different values at different places.

LEMMA 2.1. *The following holds:*

$$B(\mathbf{w}_h, q_h) = b(\gamma_h \mathbf{w}_h, q_h) = -c(\mathbf{w}_h, q_h) \quad \forall \quad \mathbf{w}_h \in \mathbf{H}_h, q_h \in L_h.$$

Proof. Since B is bilinear it suffices to show the relation holds when $\mathbf{w}_h$ is a basis function of the Raviart–Thomas space. We shall only demonstrate the relation for the vertical-edge based basis functions. The basis function $\mathbf{w}_h$ associated with the vertical edge whose midpoint is $c_{i+1/2,j}$ is supported over $Q_{i,j}$ and $Q_{i+1,j}$. It is zero in the second component and its first component is the familiar hat function with zero value on the left and right vertical edges of its support and value one on the common edge of the two rectangles above. Thus

$$\begin{aligned} -B(\mathbf{w}_h, q_h) &= (1,0)^t \int_{\partial Q_{i+1/2,j}} q_h \mathbf{n} ds \\ &= -q_h(c_{i,j})h_2 + q_h(c_{i+1,j})h_2, \end{aligned}$$

where h_2 is the height of the two rectangles involved. On the other hand,

$$\begin{aligned} c(\mathbf{w}_h, q) &= -q(c_{i,j}) \int_{Q_{i,j}} \frac{(1+0)}{h^x_{ij}} dxdy - q(c_{i+1,j}) \int_{Q_{i+1,j}} \frac{(-1+0)}{h^x_{i+1,j}} dxdy \\ &= -q(c_{i,j})h_2 + q(c_{i+1,j})h_2. \end{aligned}$$

The other cases can be handled the same way. □

We next show the coerciveness of A on the divergence free subspace of $\mathbf{H}_h$.

LEMMA 2.2. *There exists a positive constant C independent of h and $\mathbf{w}_h$ such that for all $\mathbf{w}_h \in \mathbf{H}_h$ with* $\operatorname{div} \mathbf{w}_h = 0$,

$$A(\mathbf{w}_h, \mathbf{w}_h) \geq C\|\mathbf{w}_h\|^2_{H(\mathrm{div})}$$

holds.

Proof. Write $\mathbf{w}_h = (u_h, v_h) \in \mathbf{H}_h$. Then

$$\begin{aligned} a(\mathbf{w}_h, \gamma_h \mathbf{w}_h) &= \sum_{i,j} u_h(c_{i+1/2,j}) \int_{Q_{i+1/2,j}} \tau_1(x,y) u_h(x,y)\,dxdy \\ &\quad + \sum_{i,j} v_h(c_{i,j+1/2}) \int_{Q_{i,j+1/2}} \tau_2(x,y) v_h(x,y)\,dxdy \\ &= \mathrm{I} + \mathrm{II}. \end{aligned}$$

It suffices to show that $\mathrm{I} \geq C\|u_h\|_0^2$, as II can be handled similarly. Let $Q_{ij}^- := Q_{i-1/2,j} \cap Q_{ij}$ and $Q_{ij}^+ := Q_{i+1/2,j} \cap Q_{ij}$. Then

$$\begin{aligned} \mathrm{I} &= \sum u_h(c_{i-1/2,j}) \int_{Q_{ij}^-} \tau_1(x,y) u_h(x,y)\,dxdy \\ &\quad + \sum u_h(c_{i+1/2,j}) \int_{Q_{ij}^+} \tau_1(x,y) u_h(x,y)\,dxdy \\ &= \mathrm{III} + \mathrm{IV}, \end{aligned}$$

where

$$\begin{aligned} \mathrm{III} &= \sum u_h(c_{i-1/2,j}) \int_{Q_{ij}^-} (\tau_1(x,y) - \tau_1(c_{ij})) u_h(x,y)\,dxdy \\ &\quad + \sum u_h(c_{i-1/2,j}) \tau_1(c_{ij}) \int_{Q_{ij}^-} u_h(x,y)\,dxdy \\ &= \mathrm{V} + \mathrm{VI}, \\ \mathrm{IV} &= \sum u_h(c_{i+1/2,j}) \int_{Q_{ij}^+} (\tau_1(x,y) - \tau_1(c_{ij})) u_h(x,y)\,dxdy \\ &\quad + \sum u_h(c_{i+1/2,j}) \tau_1(c_{ij}) \int_{Q_{ij}^+} u_h(x,y)\,dxdy \\ &= \mathrm{VII} + \mathrm{VIII}. \end{aligned}$$

Using the linearity of u_h in x and constant in y, we can easily derive by direct computation that

$$\mathrm{VI} + \mathrm{VIII} \geq C\|u_h\|_0^2.$$

On the other hand, by Lipschitz continuity of τ_1 and the Simpson's rule (or (2.5) below),

$$\begin{aligned} |\mathrm{V} + \mathrm{VII}| \leq Mh \sum \int_{Q_{ij}^-} |u_h(c_{i-1/2,j}) u_h(x,y)|\,dxdy \\ + \int_{Q_{ij}^+} |u_h(c_{i+1/2,j}) u_h(x,y)|\,dxdy \end{aligned}$$

$$\le Mh \sum \|u_h(c_{i-1/2,j})\|_{Q_{ij}^-} \|u_h\|_{Q_{ij}^-} + \|u_h(c_{i+1/2,j})\|_{Q_{ij}^+} \|u_h\|_{Q_{ij}^+}$$
$$\le 6Mh \sum \|u_h\|_{Q_{ij}}^2$$
$$= 6Mh\|u_h\|_0^2.$$

Thus we have

$$I \ge C\|u_h\|_0^2 - 6Mh\|u_h\|_0^2$$

and so

$$I \ge C||u_h||_0^2$$

for h sufficiently small. □

Now by Lemma 2.1, problem (1.11) becomes

$$\begin{array}{rcll} A(\mathbf{u}_h, \mathbf{w}_h) - B(\mathbf{w}_h, p_h) & = & 0 & \forall \quad \mathbf{w}_h \in \mathbf{H}_h, \\ B(\mathbf{u}_h, q_h) & = & (f, q_h) & \forall \quad q_h \in L_h. \end{array} \tag{2.1}$$

The bilinear form B is well known, and the following inf-sup condition associated with the lowest order Raviart–Thomas space can be found in [14].

LEMMA 2.3. *There exists a positive constant β independent of h such that*

$$\sup_{0 \ne \mathbf{w}_h \in \mathbf{H}_h,} \frac{B(\mathbf{w}_h, q_h)}{\|\mathbf{w}_h\|_{H(\mathrm{div})}} \ge \beta\|q_h\|_0 \quad \forall \quad q_h \in L_h. \tag{2.2}$$

Thus Lemmas 2.1–2.3 imply the uniqueness and existence of the solution of the system (1.13)–(1.14). Next we show that γ_h is a self-adjoint operator with respect to the L^2 inner product on $\mathbf{H}_h$, and it is bounded also.

LEMMA 2.4. *The following relations hold:*

$$(\gamma_h \mathbf{u}_h, \mathbf{w}_h) = (\mathbf{u}_h, \gamma_h \mathbf{w}_h) \quad \forall \quad \mathbf{u}_h, \mathbf{w}_h \in \mathbf{H}_h, \tag{2.3}$$

and there exists a positive constant C independent of h such that

$$||\gamma_h \mathbf{u}_h||_0 \le C||\mathbf{u}_h||_0 \quad \forall \quad \mathbf{u}_h \in \mathbf{H}_h. \tag{2.4}$$

Proof. Let $\mathbf{u}_h = (u_h, v_h)$ and $\mathbf{w}_h = (w_h, x_h)$. We first show that γ_h is self-adjoint. Writing out $(\gamma_h \mathbf{u}_h, \mathbf{w}_h)$ as the sum of two integrals, we see that it suffices to examine the action of γ_h on the first components (or second components). Let $u_h = a + bx$, $w_h = c + dx$ on the standard reference rectangle $Q = [0, h_1] \times [0, h_2]$ and let $(\cdot, \cdot)_Q$ denote the restriction of $(\cdot, \cdot)$ on Q and $|| \cdot ||_Q$ its induced norm. Then

$$(u_h, \gamma_h w_h)_Q = h_2 \int_0^{h_1/2} (a + bx)c dx + h_2 \int_{h_1/2}^{h_1} (a + bx)(c + dh_1) dx,$$
$$(\gamma_h u_h, w_h)_Q = h_2 \int_0^{h_1/2} a(c + dx) dx + h_2 \int_{h_1/2}^{h_1} (a + bh_1)(c + dx) dx.$$

Now their difference divided by h_2 is

$$\int_0^{h_1/2} (bc - ad) x dx + \int_{h_1/2}^{h_1} (ad - bc) h_1 dx + \int_{h_1/2}^{h_1} (bc - ad) x dx$$
$$= (bc - ad)\frac{h_1^2}{8} + (ad - bc)\frac{h_1^2}{2} + (bc - ad)\frac{3h_1^2}{8} = 0.$$

Thus, $(\mathbf{u}_h, \gamma_h \mathbf{w}_h) = (\gamma_h \mathbf{u}_h, \mathbf{w}_h)$.

The boundedness of γ_h can be proved by direct computation, but let us prove it indirectly to show a principle that will be used later. We want to show that for $E = 0, h_1$

$$\int_0^{h_2} \int_0^{h_1} |u_h(E,y)|^2 dxdy \le C||u_h||_{0,Q}^2, \tag{2.5}$$

$$\int_0^{h_2} \int_0^{h_1} |v_h(E,y)|^2 dxdy \le C||v_h||_{0,Q}^2. \tag{2.6}$$

In fact by Simpson's rule for any linear f

$$\int_0^h f^2(x)dx = \frac{h}{6}\left(f^2(0) + 4f^2\left(\frac{h}{2}\right) + f^2(h)\right)$$

and hence

$$6\int_0^h f^2(x)dx \ge hf^2(E), \qquad E = 0 \text{ or } h. \tag{2.7}$$

Thus for $u_h = a + bx$,

$$\begin{aligned}\int_0^{h_2}\int_0^{h_1} |u_h(0,y)|^2 dxdy &\le \int_0^{h_2}\left(\int_0^{h_1} h_1|u_h(0,y)|^2 dx\right) h_1^{-1} dy \\ &\le 6\int_0^{h_2}\int_0^{h_1} |u_h(x,y)|^2 dxdy \\ &= 6||u_h||_{0,Q}^2.\end{aligned}$$

Similar results hold for v_h. With these it is easy to see the boundedness of γ_h. □

Next we show a crucial approximation property of γ_h. Let us first define a discrete seminorm for $\mathbf{w}_h = (w_h, x_h) \in \mathbf{H}_h$,

$$|\mathbf{w}_h|_{1,h}^2 := \sum_{Q\in\mathcal{Q}_h} ||\nabla w_h||_{0,Q}^2 + ||\nabla x_h||_{0,Q}^2 \tag{2.8}$$

and the full norm

$$||\mathbf{w}_h||_{1,h}^2 = ||\mathbf{w}_h||_0^2 + |\mathbf{w}_h|_{1,h}^2.$$

We also use $||\mathbf{w}_h||_{1,h;Q}$ for the corresponding restriction.

LEMMA 2.5. There exists a constant C independent of h such that

$$\|(I - \gamma_h)\mathbf{w}_h\|_0 \le Ch||\mathbf{w}_h||_{1,h}, \tag{2.9}$$

$$|a(\mathbf{u}_h, (I-\gamma_h)\mathbf{w}_h)| \le Ch||\mathbf{u}_h||_{1,h}||\mathbf{w}_h||_0 \quad \forall \quad \mathbf{u}_h, \mathbf{w}_h \in \mathbf{H}_h, \tag{2.10}$$

$$|a(\mathbf{u}, (I-\gamma_h)\mathbf{w}_h)| \le Ch||\mathbf{u}||_1||\mathbf{w}_h||_0 \quad \forall \quad \mathbf{w}_h \in \mathbf{H}_h, \mathbf{u} \in \mathbf{H}^1(\Omega). \tag{2.11}$$

Proof. Using the notation in the proof of Lemma 2.4 and letting $\mathbf{w}_h = (u_h, v_h)$, we have

$$\begin{aligned}\|u_h - \gamma_h u_h\|_Q^2 &= h_2\int_0^{h_1/2} (a+bx-a)^2 dx + h_2\int_{h_1/2}^{h_1} (a+bx-a-bh_1)^2 dx \\ &= h_2\int_0^{h_1/2} b^2x^2 dx + h_2\int_{h_1/2}^{h_1} b^2(x-h_1)^2 dx = \frac{h_1^3 h_2}{12} b^2.\end{aligned}$$

Likewise, for $v_h = c + dx$,

$$\|v_h - \gamma_h v_h\|_Q^2 = \frac{h_1 h_2^3}{12} d^2.$$

So summing over $Q_{i,j}$ yields

$$\|(I - \gamma_h)\mathbf{w}_h\|_0 \leq Ch||\mathbf{w}_h||_{1,h}.$$

To prove (2.10), observe

$$\begin{aligned} a(\mathbf{u}_h, (I - \gamma_h)\mathbf{w}_h) &= a((I - \gamma_h)\mathbf{u}_h, \mathbf{w}_h) + [a(\gamma_h \mathbf{u}_h, \mathbf{w}_h) - a(\mathbf{u}_h, \gamma_h \mathbf{w}_h)] \\ &= S_1 + S_2. \end{aligned}$$

We shall show that S_1 and S_2 are bounded by the right-hand side of (2.10). First,

$$\begin{aligned} |S_1| &= |a((I - \gamma_h)\mathbf{u}_h, \mathbf{w}_h)| \\ &= |(\mathcal{K}^{-1}(I - \gamma_h)\mathbf{u}_h, \mathbf{w}_h)| \\ &= |((I - \gamma_h)\mathbf{u}_h, \mathcal{K}^{-1}\mathbf{w}_h)| \\ &\leq C||\mathcal{K}^{-1}||_\infty h||\mathbf{u}_h||_{1,h}\|\mathbf{w}_h\|_0. \end{aligned}$$

We next show how to bound S_2. Write $\mathcal{K}^{-1} = \mathrm{diag}(\tau_1(\mathbf{x}), \tau_2(\mathbf{x}))$ with $0 < t_{\min} \leq \tau_1, \tau_2 \leq t_{\max}$. With $\mathbf{u}_h = (u^1, u^2)$ and $\mathbf{v}_h = (v^1, v^2)$, we need to estimate

$$(\mathcal{K}^{-1}\mathbf{u}_h, \gamma_h \mathbf{v}_h)_Q - (\mathcal{K}^{-1}\gamma_h \mathbf{u}_h, \mathbf{v}_h)_Q = I_1 + I_2,$$

where

$$\begin{aligned} I_1 &= (\tau_1(\mathbf{x})u^1, \gamma_h v^1)_Q - (\tau_1(\mathbf{x})\gamma_h u^1, v^1)_Q, \\ I_2 &= (\tau_2(\mathbf{x})u^2, \gamma_h v^2)_Q - (\tau_2(\mathbf{x})\gamma_h u^2, v^2)_Q. \end{aligned}$$

Let $\mathbf{c} \in Q$. Now we have

$$\begin{aligned} I_1 &= (\tau_1(\mathbf{x})u^1, \gamma_h v^1)_Q - (\tau_1(\mathbf{x})\gamma_h u^1, v^1)_Q \\ &= ((\tau_1(\mathbf{x}) - \tau_1(\mathbf{c}))u^1, \gamma_h v^1)_Q - ((\tau_1(\mathbf{x}) - \tau_1(\mathbf{c}))\gamma_h u^1, v^1)_Q \\ &\quad + \tau_1(\mathbf{c})[(u^1, \gamma_h v^1)_Q - (\gamma_h u^1, v^1)_Q] \\ &= ((\tau_1(\mathbf{x}) - \tau_1(\mathbf{c}))u^1, \gamma_h v^1)_Q - ((\tau_1(\mathbf{x}) - \tau_1(\mathbf{c}))\gamma_h u^1, v^1)_Q, \end{aligned}$$

where we have used Lemma 2.4 (or the scalar version which was proved in that lemma) in deriving the last equality. Hence by the Lipschitz continuity of τ_1 and (2.4), we have

$$|I_1| \leq Mh\|u^1\|_Q\|v^1\|_Q.$$

A similar estimate holds for I_2. Summing over Q and using the Cauchy–Schwarz inequality, we obtain

$$|S_2| \leq Ch\|\mathbf{u}_h\|_0\|\mathbf{v}_h\|_0.$$

We are now ready to show the last assertion of our lemma. Let $\mathcal{E}_h$ be the familiar interpolation operator from $\mathbf{H}^1(\Omega)$ to $\mathbf{H}_h$ with $\int_e \mathbf{q} \cdot \mathbf{n} ds$, flux across edge, as its degrees of freedom [14, pp. 550–554]. Then

$$||\mathbf{q} - \mathcal{E}_h\mathbf{q}||_0 \leq Ch|\mathbf{q}|_1 \quad \forall \quad \mathbf{q} \in \mathbf{H}^1(\Omega). \tag{2.12}$$

Now

$$\begin{aligned}|a(\mathbf{u},(I-\gamma_h)\mathbf{w}_h)| &= |a(\mathbf{u}-\mathcal{E}_h\mathbf{u},(I-\gamma_h)\mathbf{w}_h)+a(\mathcal{E}_h\mathbf{u},(I-\gamma_h)\mathbf{w}_h)| \\ &\le C||\mathbf{u}-\mathcal{E}_h\mathbf{u}||_0||(I-\gamma_h)\mathbf{w}_h||_0+|a(\mathcal{E}_h\mathbf{u},(I-\gamma_h)\mathbf{w}_h)| \\ &\le Ch||\mathbf{u}||_1||\mathbf{w}_h||_0+|a(\mathcal{E}_h\mathbf{u},(I-\gamma_h)\mathbf{w}_h)|.\end{aligned}$$

It remains to estimate $a(\mathcal{E}_h\mathbf{u},(I-\gamma_h)\mathbf{w}_h)$. By (2.10),

$$|a(\mathcal{E}_h\mathbf{u},(I-\gamma_h)\mathbf{w}_h)| \le Ch||\mathcal{E}_h\mathbf{u}||_{1,h}||\mathbf{w}_h||_0.$$

Hence we will be done if we can show that

$$||\mathcal{E}_h\mathbf{u}||_{1,h} \le C||\mathbf{u}||_1.$$

Note that on any vertical edge e of Q,

$$\int_e \mathbf{u}\cdot\mathbf{n}(x,y)dy = \int_e(\mathcal{E}_h\mathbf{u})(x,y)\cdot\mathbf{n}dy = h_2(\mathcal{E}_h\mathbf{u})\cdot\mathbf{n}(x).$$

Using this and the divergence theorem, we have the partial derivative of the first component of $\mathcal{E}_h\mathbf{u}$,

$$\begin{aligned}(\mathcal{E}_h\mathbf{u})^1_x &= \frac{(\mathcal{E}_h\mathbf{u})^1(h_1)-(\mathcal{E}_h\mathbf{u})^1(0)}{h_1} \\ &= \frac{1}{h_1h_2}\int\int_Q\frac{\partial\mathbf{u}^1}{\partial x}dxdy.\end{aligned}$$

Thus

$$|(\mathcal{E}_h\mathbf{u})^1_x| \le Ch^{-1}\|\mathbf{u}^1_x\|$$

and likewise

$$|(\mathcal{E}_h\mathbf{u})^2_y| \le Ch^{-1}\|\mathbf{u}^2_y\|.$$

Since $\mathcal{E}_h\mathbf{u}$ has the form $(a+bx,c+dy)$ over Q_{ij}, we have

$$\begin{aligned}|\mathcal{E}_h\mathbf{u}|_{1,h,Q_{ij}} &\le Ch\left[|\nabla(\mathcal{E}_h\mathbf{u})^1|^2+|\nabla(\mathcal{E}_h\mathbf{u})^2|^2\right]^{1/2} \\ &= Ch\left([(\mathcal{E}_h\mathbf{u})^1_x]^2+[(\mathcal{E}_h\mathbf{u})^2_y]^2\right)^{1/2} \le C\|\mathbf{u}\|_{1,Q_{ij}},\end{aligned}$$

where we have used the quasi-regularity condition (1.4) and the fact that the partial derivatives involved are constant. Summing over Q_{ij} now completes the proof. ◻

3. Error estimates. We now prove the main theorem of this paper.

THEOREM 3.1. *Let the rectangular partition family $\{Q_{ij}\}$ of the domain Ω be quasi-regular satisfying* (1.4), *and let $\{\mathbf{u}_h,p_h\}$ be the solution of problem* (1.12) *and $\{\mathbf{u},p\}$ the solution of problem* (1.3). *Then there exists a positive constant C independent of h but dependent on $||\mathcal{K}^{-1}||_\infty$, $\|\mathbf{u}\|_1$, $\|\mathrm{div}\mathbf{u}\|_1$, and $\|p\|_1$ such that*

$$\|\mathbf{u}-\mathbf{u}_h\|_{H(\mathrm{div})}+\|p-p_h\|_0 \le Ch \tag{3.1}$$

provided that $\mathbf{u}\in\mathbf{H}^1, \mathrm{div}\,\mathbf{u}\in H^1, p\in H^1$.

Proof. Introduce the auxiliary mixed formulation to (1.3): Find $(\tilde{\mathbf{u}}_h, \tilde{p}_h) \in \mathbf{H}_h \times L_h$ such that

$$a(\tilde{\mathbf{u}}_h, \mathbf{w}_h) - B(\mathbf{w}_h, \tilde{p}_h) = 0 \qquad \forall \quad \mathbf{w}_h \in \mathbf{H}_h, \tag{3.2}$$

$$B(\tilde{\mathbf{u}}_h, q_h) = (f, q_h) \qquad \forall \quad q_h \in L_h. \tag{3.3}$$

This system has the following well-known convergence result [14]:

$$\|\mathbf{u} - \tilde{\mathbf{u}}_h\|_{H(\mathrm{div})} + \|p - \tilde{p}_h\|_0 \le Ch(\|\mathbf{u}\|_1 + \|\mathrm{div}\mathbf{u}\|_1 + \|p\|_1) \tag{3.4}$$

provided that $\mathbf{u} \in \mathbf{H}^1, \mathrm{div}\,\mathbf{u} \in H^1$, $p \in H^1$. On the other hand, we have

$$a(\mathbf{u}_h, \gamma_h \mathbf{w}_h) - B(\mathbf{w}_h, p_h) = 0 \qquad \forall \quad \mathbf{w}_h \in \mathbf{H}_h, \tag{3.5}$$

$$B(\mathbf{u}_h, q_h) = (f, q_h) \quad \forall \quad q_h \in L_h. \tag{3.6}$$

Define $\mathbf{e}_h := (\mathbf{u} - \tilde{\mathbf{u}}_h) + (\tilde{\mathbf{u}}_h - \mathbf{u}_h)$. Thus it suffices to estimate the second term on the right. Subtracting (3.6) from (3.3), we have

$$B(\tilde{\mathbf{u}}_h - \mathbf{u}_h, q_h) = 0 \quad \forall \quad q_h \in L_h. \tag{3.7}$$

Subtracting (3.5) from (3.2) yields

$$a(\tilde{\mathbf{u}}_h - \mathbf{u}_h, \gamma_h \mathbf{w}_h) + a(\tilde{\mathbf{u}}_h, (I - \gamma_h)\mathbf{w}_h) - B(\mathbf{w}_h, p_h - \tilde{p}_h) = 0. \tag{3.8}$$

Replace the $\mathbf{w}_h$ above by $\tilde{\mathbf{e}}_h := \tilde{\mathbf{u}}_h - \mathbf{u}_h$ and use (3.7) to obtain

$$a(\tilde{\mathbf{e}}_h, \gamma_h \tilde{\mathbf{e}}_h) = -a(\tilde{\mathbf{u}}_h, (I - \gamma_h)\tilde{\mathbf{e}}_h).$$

Now observe by (3.7) and (3.2) that $a(\tilde{\mathbf{u}}_h, \tilde{\mathbf{e}}_h) = 0$ and by (1.3) that $a(\mathbf{u}, \tilde{\mathbf{e}}_h) = B(\tilde{\mathbf{e}}_h, p)$ to obtain

$$\begin{aligned} a(\tilde{\mathbf{u}}_h, (I - \gamma_h)\tilde{\mathbf{e}}_h) &= a(\tilde{\mathbf{u}}_h, -\gamma_h \tilde{\mathbf{e}}_h) \\ &= a(\tilde{\mathbf{u}}_h, -\gamma_h \tilde{\mathbf{e}}_h) + a(\mathbf{u}, \tilde{\mathbf{e}}_h) - B(\tilde{\mathbf{e}}_h, p) \\ &= a(\tilde{\mathbf{u}}_h, -\gamma_h \tilde{\mathbf{e}}_h) + a(\mathbf{u}, \tilde{\mathbf{e}}_h) - B(\tilde{\mathbf{e}}_h, p - \tilde{p}_h) \\ &= a(\mathbf{u}, \gamma_h \tilde{\mathbf{e}}_h) + a(\tilde{\mathbf{u}}_h, -\gamma_h \tilde{\mathbf{e}}_h) + a(\mathbf{u}, (I - \gamma_h)\tilde{\mathbf{e}}_h) \\ &\quad - B(\tilde{\mathbf{e}}_h, p - \tilde{p}_h) \\ &= a(\mathbf{u} - \tilde{\mathbf{u}}_h, \gamma_h \tilde{\mathbf{e}}_h) + a(\mathbf{u}, (I - \gamma_h)\tilde{\mathbf{e}}_h) \\ &\quad - B(\tilde{\mathbf{e}}_h, p - \tilde{p}_h). \end{aligned}$$

Hence, we have the error equation

$$a(\tilde{\mathbf{e}}_h, \gamma_h \tilde{\mathbf{e}}_h) = -a(\mathbf{u} - \tilde{\mathbf{u}}_h, \gamma_h \tilde{\mathbf{e}}_h) - a(\mathbf{u}, (I - \gamma_h)\tilde{\mathbf{e}}_h) + B(\tilde{\mathbf{e}}_h, p - \tilde{p}_h). \tag{3.9}$$

By Lemma 2.2, (2.11), the boundedness of B, and (3.4),

$$\begin{aligned} \alpha \|\tilde{\mathbf{e}}_h\|^2_{H(\mathrm{div})} &\le C\|\mathbf{u} - \tilde{\mathbf{u}}_h\|_0 \|\tilde{\mathbf{e}}_h\|_0 + Ch\|\tilde{\mathbf{e}}_h\|_0 ||\mathbf{u}||_1 + Ch\|\tilde{\mathbf{e}}_h\|_{H(\mathrm{div})} \\ &= C\|\mathbf{u} - \tilde{\mathbf{u}}_h\|_0 \|\tilde{\mathbf{e}}_h\|_{H(\mathrm{div})} + Ch\|\tilde{\mathbf{e}}_h\|_{H(\mathrm{div})} ||\mathbf{u}||_1 \\ &\quad + Ch\|\tilde{\mathbf{e}}_h\|_{H(\mathrm{div})}, \end{aligned}$$

where C is independent of h but dependent on $||\mathcal{K}^{-1}||_\infty$, $\|\mathbf{u}\|_1$, $\|\mathrm{div}\mathbf{u}\|_1$, and $\|p\|_1$. Note that we have used (3.4) to estimate the last term. The first term can be further estimated by (3.4) to extract a power of h and hence

$$\|\tilde{\mathbf{e}}_h\|_{H(\mathrm{div})} \le Ch.$$

An application of the triangle inequality completes the proof for the velocity. The error in the pressure is estimated by invoking the inf-sup condition. ∎

It should be clear that Theorem 3.1 still holds in three dimensions. The reader is referred to Duncan and Jones [9] for computational results of the Dirichlet problem in three dimensions with identity mobility matrix. In actual computation, one needs quadratures to evaluate the bilinear forms in the above theorem. Hence the actual forms are perturbations of the exact forms. This type of analysis has been done in the finite element literature by Shen [16], and we will not repeat it here. We have also obtained the convergence results for the lowest order Raviart–Thomas space on triangular grids in Chou, Kwak, and Vassilevski [8].

REMARK 3.1. *Our results in this paper do not cover the cases of nonrectangular quadrilateral grids (logically rectangular grids) and nondiagonal tensor permeability coefficients, which are of importance in some applications, particularly subsurface reservoir flow. (The analysis in* [8] *carries over to the nondiagonal tensor case, but it is only for triangular elements.) We are currently pursuing these topics.*

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